THESIS

Subject THE GENERATION OF CONIC SECTIONS BY PROJECTIVE RANGES AND PENCILS:---A STUDY OF THE NATURE OF THE CONIC B Y MEANS OF THE NATURE OF THE PROJECTIVE RELATION INVOLVED IN ITS GENERATION.

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THE GENERATION OF CONIC SECTIONS BY PROJECTIVE RANGES AND PENCILS:---A STUDY OF THE NATURE OF THE CONIC BY MEANS OF THE NATURE OF THE PROJECTIVE RELATION INVOLVED IN ITS GENERATION.

A thesis submitted to the faculty of the Graduate School of The University of Minnesota by

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in partial fulfillment of the requirements for the degree of Master of Arts.

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OUTLINE.

I. Historical sketch of conics. (#1)

1. Divided into 3 classes by Menaechmus, 4th century B. C.

2. Named by Apollonius, 3d century B. C.

3. Shown to be of practical use by Kepler, 17th century A. D.

4. Projection and parallel lines meeting at infinity, Desargues and Pascal.

5. Modern Writers: Steiner; Chasles; Von Staudt.

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3. Pencils.

a. Special correspondence.

(1) Exceptional rays not coinciding: line-pair. (#53)

(2) Exceptional rays coinciding: point-pair. (#54)

b. Special position.

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(a) Line-pair.

(1) Real.

(2) Imaginary.

(b) Line-pair-point.

REPORT

of

COMMITTEE ON THESIS

THE undersigned, acting as a committee of the Graduate School, have read the accompanying thesis submitted by Miss Mary Ella Hartwell. for the degree of Master of Arts. They approve it as a thesis meeting the requirements of the Graduate School of the University of Minnesota, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts.

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May 26 1914

I. HISTORICAL SKETCH.

#1. While the generation of conics by means of two projective forms has been set forth more explicitly by Steiner and Chasles than by any previous writers, yet some of the principles involved in their work were known as early as the third or fourth century B. C. It was in the fourth century that conics were discussed for the first time by Menaechmus, who divided them into three classes and investigated their properties to a very limited degree; he seems to have regarded them, not as plain loci, but as curves drawn on the surface of a cone. In the next century Apollonius carried these investigations still further, and also gave the names ellipse, parabola, and hyperbola to the three kinds of conics. Cremona says, in the preface to the first edition of his "Elements of Projective Geometry", that the projectivity of pencils formed by joining two fixed points on a conic to a variable point on the same had already been proved, in other words, by Apollonius. But with his death the advance in this line of study ceased, and it was not until the beginning of the seventeenth century that any further progress was made. At the close of the sixteenth century interest in the study of conics was revived slightly by Kepler, who showed one practical use for them in tracing the movements of the heavenly bodies. He also introduced the principle of continuity, and showed that a parabola has one focus at infinity and that lines radiating from this point are parallel. From Kepler's doctrine of continuity Desargues developed the theory that all parallel straight lines meet in an infinitely distant point. And he, together with Newton, considered the asymptotes of the hyperbola as tangents whose points of contact lie at an infinite distance. Pascal and Desargues introduced the method of projection and conceived the treatment of the conic sections as projections of circles. The geometric forms of one dimension, namely, the range of points and the flat pencil, are found, the names excepted, in Desargues and the other geometers of his time. Among those of later times who have contributed much to this division of mathematics are MacLaurin, Poncelet, Chasles, Von Staudt, and Steiner, who has given us a more complete discussion on the generation of conics by projective forms than any of the others.

II. FUNDAMENTAL FACTS.

#2. The generation of conics by means of two projective forms is based on the two following fundamental theorems. For proofs of them see Steiner,[1] Chasles,[2] and Cremona.[3] "If two (non-collinear) ranges lying in the same plane are projective (but not in perspective) the envelope of the straight lines joining pairs of corresponding points is a conic, i.e., the straight lines all touch a conic. This conic touches the bases of the two ranges at the points which correspond in these respectively to the point of intersection of their bases." Also, "if two (non-concentric) pencils lying in the same plane are projective with one another (but not in perspective) the locus of the points of intersection of pairs of corresponding rays is a conic passing through the centers of the two pencils; and the tangents to the locus at these points are the rays which correspond in the two pencils respectively to the straight line which joins the two centers." These are the fundamental theorems upon which is based the following study of the nature of the conic as determined by the nature of the generating ranges or pencils.

(1) Vorlesungen uber Syntetische Geometrie. II, #22.

(2) Aperçu historique sur l'origine et le developpement des methodes en géomètrie. Notes XV & XVI.

(3) Elements of Projective Geometry. #150.

#3. As an aid to this study, the characteristics of the three kinds of conic sections, namely, the ellipse, parabola, and hyperbola, must be noted so that it will be possible to recognize them. And since conics are sections of a cone made by a cutting plane, the number of points at infinity can easily be determined. The ellipse is the section of a cone made by a plane which cuts all of the elements in finite space; therefore it has no points at infinity. The parabola is formed when this cutting plane is parallel to one element; therefore it has one, and only one, point at infinity. The hyperbola, being made by a plane cutting both nappes of the cone, has two points at infinity. Therefore, an ellipse can have no tangents touching it at infinity; a parabola, one; and an hyperbola, two. Thus it is possible to determine whether the generated conic is an ellipse, parabola, or hyperbola by finding either the number of tangents or the number of points it has at infinity.

III. THE CONIC AS THE ENVELOPE OF CONNECTORS OF CORRESPONDING POINTS.

#4. Since a conic may be looked upon in two ways, namely, as the envelope of the connectors of corresponding points of two projective ranges or as the locus of the point of intersection of corresponding rays of two projective pencils, the subject naturally divides itself into two distinct parts. There is also a third division, when the conic degenerates or, in other words, when the projective relation is of a special kind or when the relative positions of the ranges or pencils are of a special kind.

#5. In the first case, in order to examine the conic which is regarded as the envelope of the connectors of corresponding points of two projective ranges it is necessary to be able to locate the points of contact of these connectors which will be the tangents to the curve. Steiner (1) gives this method: suppose there are two ranges, U and U_1 (Fig. I) intersecting at e of U, and f_1 of U_1. Then their contact points are f and e_1 respectively, since they correspond to the intersection of the two bases. Let x, x_1 be any other pair of corresponding points. By joining x and x_1 the triangle e x x_1 is formed, which is a circumscribed triangle to the conic with the contact points of the two tangents known. When each of the vertices is connected with the contact point of the opposite side, the three lines are concurrent. This follows from a special case of Brianchon's theorem regarding a circumscribed hexagon when the hexagon has

(1) Steiner: loc. cet. #21, n r. 67.

degenerated to a triangle. Thus, if $f\,x_{,}$ and $e_{,}\,x$ are drawn, intersecting in M, and e M drawn, meeting $x\,x_{,}$ in B, then B is the contact point of $x\,x_{,}$.

#6. Another way of lòoking at it is this: B (Fig.I) is the harmonic conjugate of A, the intersection of $x\,x_{,}$ with $f\,e_{,}$, the line joining the contact points, with respect to x and $x_{,}$, since $e\,f\,M\,e_{,}$ is a complete quadrangle of which two sides go through x (viz., e f and $e_{,}$ M), two through $x_{,}$ (viz., f M and $ee_{,}$), one through B (viz., e M), and one thsough A (viz., $f\,e_{,}$), and therefore the points $A\,B\,x\,x_{,}$ are four harmonic points.

Thus the contact point on any connector of corresponding points of two projective ranges can be found provided the contact points on the ranges themselves are known. But these can always be located by means of the given projective relation by which, also, all corresponding points can be constructed.

#7. In the study of the projective relations, it is found that all ranges are divided into two classes, namely, those with vanishing points and those without vanishing points. The ranges without vanishing points are those whose infinitely distant points correspond to each other. In this case it is well known that corresponding segments of the two ranges are proportional.[1] The ranges are said to be similar. Let U and U_1 (Fig. II) be two such ranges with A, A_1, and B, B_1 any two pairs of corresponding points. Since the ratio of any two corresponding segments is a constant, $\frac{A_1B_1}{A\ B}$ = a constant.

If A and A_1 are chosen as origins and $A\ B = x$ and $A_1B_1 = x_1$, and the constant = K, then $\frac{x_1}{x} = K$, or $x_1 = Kx$. This gives a perfectly general relation for all similar projective ranges as A, A_1 and B, B_1 are any pairs of points. So, also, it holds true, no matter which pair of points is chosen for origins. Thus the projective relation for any two ranges without vanishing points is given by the equation $x_1 = K\ x$, where x and x_1 may be called the co-ordinates of corresponding points.

#8. In the other case, where the ranges have vanishing points, the projective relation can always be reduced to the form $x_1 = \frac{1}{x}$. For example, suppose U and U_1 (Fig. III) are two projective ranges with J^∞ and I_1^∞, the infinitely distant points, and J_1 and I their respective corresponding points or, in other words, the vanishing points of the ranges. Let A, A_1 be any pair of corresponding points on U and U_1 respectively. The product $I\ A \cdot J_1A_1$ = a constant[2] for all pairs of corresponding points. Let the vanishing points be

(1) Cremona, loc. cit. #100.

(2) Cremona, loc. cet. #74.

chosen as origins and let $I\,A = x$ and $J_1\,A_1 = x_1$, and let the constant be K^2. Then the equation becomes $x \cdot x_1 = K^2$ or $x_1 = \frac{K^2}{x}$. This equation gives the projective relation between any two projective ranges with vanishing points.

#9. This relation can be reduced to the simpler form $x_1 = \frac{1}{x}$, if the unit of measurement is properly chosen. For example, since x_1 means the distance from J_1 to A_1 and x the distance from I to A, it is easily seen that there is a pair of points H, H_1 (Fig. IV) whose distance from I and J_1 will be equal, that is, $I\,H = J_1 H_1$. Therefore $x \cdot x_1 = I\,H \cdot J_1 H_1 = (I\,H)^2 = (J_1 H_1)^2 = K^2$. Then $I\,H = J_1 H_1 = K$; and since there may be a negative square root of K^2, namely, $-K$, then there will be another pair of points G, G_1 where $I\,G = J_1 G_1 = -K$; in other words, there is a pair of points on each side of the vanishing points that are the same distance from them, i.e., $I\,H = I\,G$. These points are called " power points" by Steiner (1). Then, if in measuring the distances of corresponding points this distance I H is used as the unit, $\pm K$ becomes ± 1 and therefore $K^2 = 1$, and the projective relation becomes $x_1 = \frac{1}{x}$.

Conversely, let there be given four points on one line, each represented by a co-ordinate x with respect to an origin O

(1) Steiner: loc. cet. #12, nr. 25.

(Fig. V), and four points on another line, each represented by a co-ordinate x_1 with respect to an origin O_1, determined from the first four by the felation $x_1 = \frac{1}{x}$; then the relation between the ranges is a projective one. For, let a_1, a_2, a_3, and a_4 be the co-ordinates of the four points on the first line U ; and let b_1, b_2, b_3, and b_4 be the co-ordinates of the four points on the second line U_1. Then[1] $b_1 = \frac{1}{a_1}$, $b_2 = \frac{1}{a_2}$, $b_3 = \frac{1}{a_3}$, and $b_4 = \frac{1}{a_4}$. The anharmonic ratio of the first four is

$$\frac{(a_2-a_1)}{(a_3-a_2)} : \frac{(a_4-a_1)}{(a_4-a_2)} = \frac{(a_2-a_1)}{(a_3-a_2)}\,\frac{(a_4-a_2)}{(a_4-a_1)}.$$

The anharmonic of the second four is

$$\frac{(b_2-b_1)}{(b_3-b_2)} : \frac{(b_4-b_1)}{(b_4-b_2)} = \frac{(b_2-b_1)}{(b_3-b_2)}\,\frac{(b_4-b_2)}{(b_4-b_1)}$$

which is equal to $\dfrac{\left(\frac{1}{a_3}-\frac{1}{a_1}\right)\left(\frac{1}{a_4}-\frac{1}{a_2}\right)}{\left(\frac{1}{a_3}-\frac{1}{a_2}\right)\left(\frac{1}{a_4}-\frac{1}{a_1}\right)}$

or $\dfrac{\frac{(a_1-a_3)}{a_1a_3}\cdot\frac{(a_2-a_4)}{a_2a_4}}{\frac{(a_2-a_3)}{a_3a_2}\cdot\frac{(a_1-a_4)}{a_1a_4}} = \dfrac{(a_1-a_3)\cdot(a_2-a_4)}{(a_2-a_3)\,(a_1-a_4)}$

Or, by changing signs, $\frac{(a_2-a_1)}{(a_3-a_2)} : \frac{(a_4-a_1)}{(a_4-a_2)}$, which is equal to the

(1) Emch, Introduction to Projective Geometry: #2.

anharmonic ratio of the first four. Therefore the ranges are projective.

#10. In the investigation of the conic generated by similar ranges, that is by ranges whose projective relation is $x_1 = Kx$ or whose points at infinity correspond to each other, it is readily seen that there must be one line which is tangent to it at infinity, since the connectors of corresponding points are tangents to the conic and one of those connectors will be the line at infinity. And as there is but one of these lines at infinity, it is readily seen that the conic is a parabola, since the parabola is the only one of the conic sections which has the line at infinity for a tangent. Steiner (1) says, "The line at infinity is, as we know, tangent to the parabola. And any two tangents taken as carriers of two generating ranges will cut it in infinitely distant points which must be corresponding points. Two ranges whose points at infinity correspond are necessarily similar; therefore we see that a parabola can be formed only from two similar ranges and will always be formed, as long as they are not in perspective." If they are in perspective the conic will be one of the degenerate forms treated in #50. Therefore, in conclusion, two similar ranges, not in perspective, will always generate a parabola.

(1) Steiner: loc. cet. #26, nr. 78.

#11. As it has been proved that projective ranges without vanishing points always generate a parabola, the ranges with vanishing points or those whose projective relation is $x_{,} = \frac{1}{x}$ (#9) will next be examined. The first to be considered will be ranges on parallel lines and second, those on lines intersecting at different angles, in which case they will be studied with respect to the positions of their origins relative to the intersection point of the given lines.

#12. First, it is seen that the nature of the conic generated by ranges on parallel lines depends upon whether the ranges are equally or unequally directed. If x moves to the right when $x_{,}$ moves to the right, and vice versa, the ranges are said to be equally directed. But if x moves to the right when $x_{,}$ moves to the left, and vice versa, they are said to be unequally directed. If like directions on the two parallel lines are taken as the positive directions, the projective relation determined by $x_{,} = \frac{1}{x}$ is such that x and $x_{,}$ move in opposite directions. That is, the ranges are unequally directed.

In this case an ellipse is generated. For let there be two parallel ranges U and $U_{,}$ (Fig. VI) having their positive directions to the right from I and $J_{,}$ the respective origins. These points are also the vanishing points and therefore the contact points of the two fanges, sånce the ranges intersect at infinity. Then taking points H, G, B, etc. on U, the corresponding points of $U_{,}$, i.e. $H_{,}$, $G_{,}$, $B_{,}$, etc. can easily be found from the relation $x_{,} = \frac{1}{x}$. Suppose H and $H_{,}$ are a pair of power points referred to in #9, then, since $I H = J_{,} H_{,}$ and since U and $U_{,}$ are parallel, $H H_{,}$ is parallel to the contact line $I J_{,}$ and therefore intersects it at infinity. This puts the contact point C of the connector $H H_{,}$ midway between H and $H_{,}$. For Cremona[(1)] says, "If in a harmonic range A B C D the point C lies midway between the two conjugates A and B, then the fourth point D lies at an infinite distance; and conversely, if one of the points D lies at infinity, its conjugate C is the point midway between the two others, A and B." Now if X, a point of U, moves from H to I, its correspondent $X_{,}$ moves from $H_{,}$ to $I_{,}+\infty$. The intersection of $X X_{,}$ with $I J_{,}$ moves in from infinity toward X and at the same time the contact point moves from its central position toward X and therefore is always finite. The same thing holds true as $V_{,}$, a point of $U_{,}$, moves from $H_{,}$ to $J_{,}$ while V goes from H to $J +\infty$. Similar statements are true regarding the points on the negative sides of the

(1) Cremona: loc. cet. #59.

origins. Thus all of the contact points are finite and the curve is an ellipse.

#13. If opposite directions on the two parallel lines are taken as the positive directions for measuring x and x_1, the projective relation determined by $x_1 = \frac{1}{x}$ is such that x and x_1 move in the same directions on the parallel lines. That is, the ranges are equally directed.

In this case the conic generated is an hyperbola. For suppose there are two parallel ranges U and U_1 (Fig. VII) with I and J_1 their respective origins. Let the positive directions be respectively to the right of J_1 and to the left of I. Then if B is a point on U at the right of I, it follows from the projective relation $x_1 = +\frac{1}{x}$, that the corresponding point B_1 will be to the left of J_1, and thus the points between I and $-\infty$ on U (that is, the points to the right of I) correspond to the points between $-\infty$ and J_1 on U_1 (i. e., the points to the left of J_1). Therefore all the connecters will intersect I J_1 on their internal segments (i. e., segments intercepted by the given parallel lines), and therefore their contact points are on their external segments. But (from #6) the point of intersection of a connector with the line joining the vanishing points is the harmonic conjugate of the contact point with respect to the pair of corresponding points; therefore (from #12) when the connector H $H_{,}$ is bisected by the line I $J_{,}$, the contact point will be at infinity and the line will be a tangent to the curve at infinity. But, since there are two pairs of points, $H, H_{,}$ and $G, G_{,}$ (i. e., the power points) which are equally distant from I and $J_{,}$, there are two such connectors and thus there are two lines tangent to the curve at infinity, which proves that the curve is an hyperbola, of which these two special connectors are the asymptotes. Thus it is seen that ranges whose points move in opposite directions on parallel lines generate an ellipse and those whose points move in the same direction generate an hyperbola.

#14. This can be stated in a different way. That is, when two projective ranges are on parallel lines, if the contact points of the ranges are on the same side of a connector of corresponding points the curve generated is an ellipse; if the contact points are on opposite sides of a connector of corresponding points the ranges generate an hyperbola.

#15. There is an interesting special case of the generation of conics by ranges on parallel lines when the ranges are unequally directed and when the vanishing points I and $J_,$ are so placed on the lines that the line I $J_,$ is perpendicular to the two axes, and equal to twice the distance (I H) of the power points from the origins. Suppose H, $H_,$ and G, $G_,$ (Fig. VIII) are the power points. Then H $H_,$ G $G_,$ is a square and M and N, the contact points on H $H_,$ and G $G_,$ respectively, are the middle points of these lines, for these lines intersect I $J_,$ at infinity (from #12). Also I and $J_,$, the contact points of U and $U_,$., are the middle points of H G and $H_,$ $G_,$ respectively. Therefore the conic is a circle, because I $J_,$ = M N and these segments are diameters of the conic since they join contact points of parallel tangents.[1]

The circle can be generated under no other conditions when the axes are parallel for I $J_,$ will always be the chord of contact of two parallel tangents and therefore a diameter. But in circles diameters are perpendicular to the tangents drawn at their extremities. Thus if I $J_,$ is not perpendicular the conic cannot be a circle.

#16. There is another interesting special case when the ranges are equally directed and each of the equal distances I H, $J_,$ $H_,$, I G, $J_,$ $G_,$ (where H, $H_,$ and G, $G_,$ are the power points) is half the

(1) Cremona: loc. cet. # 282.

length of $I J_{\prime}$ (Fig. IX). Then $H G_{\prime} H_{\prime} G$ is a rhombus, and the diagonals $H H_{\prime}$ and $G G_{\prime}$ are therefore perpendicular to each other. As these connectors are the asymptotes of the conic (from #13), the curve is an equilateral hyperbola.

#17. In determining the nature of the conic generated by ranges on non-parallel lines, the angle between the lines and the position of the vanishing points relative to the intersection and the positive directions as chosen on the lines will have to be considered.

First, let the ranges be in such a position that both of the vanishing points are at the intersection of the lines (Fig. X). In this case the intersection $I J_{\prime}$ corresponds to $I_{\prime}^{\infty}$ and J^{∞} at infinity on $U_{\prime}$ and U respectively. Therefore the contact points lie at infinity, since they correspond to the intersection. This shows that the conic has two lines tangent at infinity and therefore is an hyperbola of which the two lines U and $U_{\prime}$ are the asymptotes. This is true no matter in which directions the points of the ranges move and no matter at what angle the lines are drawn. When the lines are perpendicular the hyperbola will be equilateral.

#18. Now suppose the ranges are such that but one of the vanishing points is at the intersection of the lines U and $U_{\prime}$ (Fig.XI).

That is, let I, the vanishing point of U, be at the intersection; its correspondent $I_{/}$, the contact point of $U_{/}$, is at infinity. Then since the ellipse can have no points at infinity, the conic generated in this case cannot be an ellipse; and, as has already been proved, the parabola can be generated only by ranges whose points at infinity are correspondents; therefore the conic cannot be a parabola, and so it must be an hyperbola. This is a sufficient determination of the nature of the conic, but when an hyperbola is generated, it is always of interest to find the asymptotes. Since $U_{/}$ is tangent to the curve at infinity, it is one of the asymptotes. Now if the point of intersection, thought of as belonging to $U_{/}$, is called $A_{/}$, then, by the projective relation $x_{/} = \frac{k}{x}$, its correspondent on U, called A, can be found. This point A will be the contact point of U. If the contact line $A\ I_{/}^{\infty}$ is drawn (parallel to $U_{/}$ of course) and on U a point B is taken on the other side of A from I such that B A shall be equal to A I, and if its correspondent $B_{/}$ on $U_{/}$ be found and $B\ B_{/}$ drawn, the internal segment of $B\ B_{/}$ will be bisected at P by $A\ I_{/}^{\infty}$, because $A\ I_{/}^{\infty}$ and $U_{/}$ are parallel, and therefore the contact point on $B\ B_{/}$ will lie at infinity (from #12); and as there is but one point B at a distance of I A from A, since the other point is I itself, there can be but one connector tangent to the curve at infinity other than the line U_1 . Therefore these two lines, U_1 and $B B_1$, are the asymptotes. The choice of positive and negative directions on the lines does not affect the proof as $B B_1$ will be bisected by $A I_1^{\infty}$ no matter upon which side of J_1 it may be.

#19. The most general case of a conic generated by two projective ranges occurs when neither of the vanishing points is at the intersection of the bases of the given ranges. Suppose that the respective vanishing points, I and J_1 (Fig. XII) are both on the positive side of the intersection which is called B of range U and A_1 of range U_1 . Then by means of the projective relation the correspondents B_1 and A can be found. These are the contact points of the two lines. Here there are two possibilities, namely, both A and B_1 will lie on the positive side of the intersection or both on the negative side, according to the distance of the power points from I and J_1. The first of these possibilities is shown in Fig. XII, the second in Fig. XIII. But A can never be on the positive side while B_1 lies on the negative, for the points between I and A correspond to those between A_1 and $I_1^{-\infty}$, and therefore when B lies between I and A (Fig. XIII), so must B_1 lie between A_1 and $I_1^{-\infty}$; while if A_1 is between

B_1 and $I_1^{-\infty}$, A (Fig. XII) must be between B and I. Thus the points A and B_1 lie either both on the positive side or both on the negative side of the intersection. I cannot lie between B and A because A_1 is on the negative side of J_1 and therefore A must be on the negative side of I.

#20. These two cases must be examined separately in determining the nature of the conic generated. The first to be considered will be the case where both A and B_1 lie on the positive side of the intersection. (Fig. XIV). Then if a line U_2 be drawn through I parallel to U_1 it will be a tangent to the conic as it will be the connector I I_1^{∞}. Now since "any number of tangents to a conic determine on a pair of fixed tangents two projective ranges,"(1) the connectors of the corresponding points of U and U_1 will determine on U_2 a range of points projective with the range on U_1. The lines joining the corresponding points of U_2 and U_1 will generate the same conic as those joining the corresponding points of U_1 and U, since the tangents will be identical in both cases. Then the nature of the conic generated by the ranges U and U_1 can be determined by the examination of the conic generated by the ranges U_1 and U_2 which are parallel.

Now I considered as a point of U_2 corresponds to A_1 of U_1

(1) Cremona: loc. cet. #149.

and in the diagram is marked A_2, for, since $A_1 I$ is one of the given tangents to the conic it must be a connector of two corresponding points of the ranges U_1 and U_2; and A is the contact point of U which is the connector of A_1 and A_2. But as A lies between A_1 and A_2, the intersection of A_1A_2 with the contact line of the two new ranges, U_1 and U_2, will lie on the external segment of the connector (from #6). Let this intersection point be called R. But B_1 is the contact point of U_1; therefore $B_1 R$ is the contact line of U_1 and U_2. But since R is on the external segment of A_1A_2, this line B_1R cuts U_2 in a point M which is on the same side of A_1A_2 as B_1 is. Or, in other words, the contact points M and B_1 of the two parallel ranges U_1 and U_2 lie on the same side of the connector A_1A_2. Therefore (from #14) the conic is an ellipse.

#21. When the contact line lies on the negative side of the intersection the ranges generate an hyperbola. For suppose the two ranges U and U_1 (Fig. XV) have the vanishing points I and J_1 respectively, intersect at the points B or A_1, and have the contact points A and B_1. In this case the contact points of $I I_1^{\infty}$ and $J^{\infty}J_1$ will lie within the angle $(I B J_1)$ since their intersections with the contact line A B_1 are on the outside of the angle. Therefore the conic will have two branches, one within the angle (IBJ_{1}) and the other within the angle $(B_{1}BA)$ since B_1 and A are the contact points of U_1 and U. But the conic of two branches is the hyperbola.

#22. Now if the ranges U and U_1 (Fig. XVI) are taken in such a position that the intersection B_1A_1 is on the negative side of the vanishing point J_1 and on the positive side of I, then the contact points A and B_1 will be on the opposite sides of the vanishing points from the intersection. For since B is on the positive part of U, B_1 must be on the positive segment of U_1, and likewise A and A_1 must both be on the negative segments because $x_1 = \frac{+1}{x}$. Thus when the intersection lies on segments of unlike sign with reference to the origins, the vanishing points will always lie between it and the contact line. Then the contact point of $I\,I_1^{\infty}$ which is the harmonic conjugate of M with respect to I and I_1^{∞}, will be at N which is as far from I as M is, but on the other side. Therefore the conic touches the sides of the angle $(Z\ I\ A)$ at two points A and N. Likewise the conic will touch the sides of the angle $(W\ J_1B_1)$ at two points Q and B_1. Therefore the curve must have two branches and is an hyperbola.

#23. Thus when neither of the two ranges on non-parallel lines has its vanishing point at the intersection the nature of the generated conic is determined by the position of the contact line with reference to the vanishing points and the intersection. When the contact line lies between the vanishing points and the intersection the conic is an ellipse, but when either the vanishing points or the intersection lies between the contact line and the other, the conic is an hyperbola. As the size of the angle between the bases of the two ranges has nothing whatever to do with the position of the contact line it is easily seen that the ranges may be revolved about their intersection without changing the nature of the conic. But when either range is shoved along its base, then the nature of the conic is likely to be changed.

#24. In a general way, if any two lines are given with three points of one range and the three corresponding points of the other, the nature of the generated conic can be determined, by locating the contact points and the vanishing points. Suppose U and $U_,$ (Fig. XVII) are the two lines with the given points A, B, and C, on U and their correspondents $A_,$, $B_,$, and $C_,$, on $U_,$. Call the intersection D on U, and $E_,$ on $U_,$. Then draw $C_, A$ meeting $C A_,$ in R, also $B_, A$

meeting B A_1 in S. The points where R. S touch U_1 and U correspond to the intersection[1] and are D_1 and E respectively. Therefore these points are the contact points of U_1 and U and E D_1 the contact line. Now in ofder to find the vanishing points I and J_1 draw A I_1^{∞} (that is, a line through A parallel to U_1) cutting R S in P ; then draw A_1 P meeting U in I which will be the vanishing point of U. Next draw B_1 J^{∞} (i. e., a line through B_1 parallel to U) intersecting R S in T and draw B T which will cut U_1 in J_1 , the vanishing point of U_1 . Then, in this particular diagram, since I and J_1 are between the contact line and the intersection (from # 22), the ranges generate an hyperbola.

#25. The following is a summary of the results already obtained. The generated conic is---

a parabola, when the projective ranges are similar:

an ellipse, (a) when the ranges are unequally directed on parallel lines, (b) when the contact line lies between the vanishing points and the intersection of the ranges on two non-parallel lines at any angle:

an hyperbola, (a) when the ranges are equally directed on parallel lines, (b) when either both or only one of the vanishing

(1) Cremona: loc. cet. #85.

points is at the intersection, (c) when the contact line does not lie between the intersection and the vanishing points.

IV. THE CONIC CONSIDERED AS THE LOCUS OF INTERSECTIONS OF CORRESPONDING RAYS.

#26. In the examination of the conic, when it is considered as the locus of the intersections of corresponding rays of two projective pencils, it must first be noted as a general fact that the nature of the generated curve may be discovered by determining the number of pairs of parallel corresponding rays. Because, since parallel lines are considered as meeting at infinity, such rays will intersect at an infinite distance and so locate the infinite points of the conic. Therefore if there are two pairs of parallel rays, and, as is well known, there can never be more than two, the curve will have two points at infinity and will be an hyperbola; if there is but one pair of such rays, the curve will have one infinite point and will be a parabola; and if there are no parallel rays, there will be no infinite points and therefore the conic will be an ellipse.

#27. Projective pencils may be divided into two classes as follows:---

If when a variable ray of one pencil rotates in a clockwise sense, the corresponding ray of the other pencil rotates in a counter-clockwise sense, the two pencils are said to be oppositely directed. But when a variable ray of the one rotates in the same sense (clockwise or counter-clockwise) as the corresponding ray of the other, the pencils are said to be equally directed.

#28. Since in the case of two oppositely directed pencils there will always be two pairs of parallel corresponding rays, then such pencils will always generate an hyperbola.

#29. If the pencils are equal (i. e., if the angle between any two rays of one pencil equals the angle between the corresponding rays of the other) as well as oppositely directed, the hyperbola will be equilateral. For let O and $O_{/}$ (Fig. XVIII) be the centers of two such pencils and suppose the $O_{/}$ pencil to be shifted without rotation until it is concentric with O . Then the internal and external bisectors, d and e, of the angle between any two corresponding rays, a and $a_{/}$ will be double rays of the concentric pencils, since either one of them makes equal angles with the rays a and $a_{/}$, also with b and $b_{/}$, and so may be regarded as belonging to each pencil. But these rays give the direction of the infinite points of the curve generated by the two given pencils, and therefore they are parallel to the asymptotes. And since they are the internal and external bisectors of an angle they are perpendicular to each other and the hyperbola is equilateral. Thus, when the pencils are oppositely directed they may be revolved about their centers in any way and the conic will always remain an hyperbola except in one position when it degenerates. That position is when two corresponding rays coincide or, in other words, when the pencils are in perspective (see #56.).

#30. But in the case of equally directed pencils, since they may have two pairs, one pair, or no pair of parallel rays, the curve may be any one of the three, an hyperbola, a parabola, or an ellipse. If the centers are both in finite space the nature of the conic can be determined by moving the pencil O (Fig. XIX) without rotation until it is concentric with the pencil O_1. Then with any radius construct a circle through O_1 cutting the six rays a, b, c, a_1, b_1, c_1 in the points A, B, C, A_1, B_1, C_1 respectively. Join A_1 C and A C_1 meeting in R, and B_1 C and B C_1 meeting in S. Draw R S intersecting the circle in M and N. Then O_1 M and O_1 N will be the double rays of the two concentric projective pencils [1] or will give the direction of the parallel rays which give the points at infinity on the conic generated by the two given pencils. Thus the two

(1) Cremona: loc. cet. #206.

pencils chosen in Fig. XIX. generate an hyperbola because R S meets the circle in two points. If the line R S is tangent to the circle, there is but one pair of parallel rays and the conic is a parabola. If R S lies entirely outside of the circle, there are no parallel rays and the pencils generate an ellipse. Thus by this method the nature of the generated conic can always be determined when three rays of each pencil are given and the centers are in finite space.

#31. There is again the special case when the pencils are equal as well as equally directed, in which case the conic is a circle. Let O and O_1 (Fig. XX) be two such pencils with a and a_1 meeting in A, b and b_1 meeting in B, and c and c_1 meeting in C. Then, since the angles (A O B) and (A O_1 B) are equal, the angles at A and B are equal. Likewise the angle between any two corresponding rays will always equal the angle at A. Therefore the locus of these intersections must be a circle by a well known theorem in elementary geometry.

#32. If both the centers of the equally directed pencils are at infinity, that is, if each pencil consists of parallel rays which are thought of as rotating about their centers at infinity in the same direction (clockwise or counter-clockwise), then an hyperbola is generated for the following reasons:--- The connector of the centers O and $O_{/}$ is the line at infinity, and since the ray of either pencil that corresponds to O $O_{/}$, considered as a ray of the other, is a tangent to the conic at O or $O_{/}$, there will be two lines tangent at infinity. This determines the curve to be an hyperbola of which the two lines are the asymptotes. Another way of seeing that the conic is an hyperbola is this. Since O and $O_{/}$ are two distinct points at infinity and as the conic must always pass through the centers of the pencils, the curve will be an hyperbola, as that is the only conic which has two points at infinity.

#33. When the center of one pencil is in finite space and the center of the other is a point at infinity, there may be two distinct results, because the tangent to the conic at $O_{/}$, the center at infinity, may be a line in finite space or the line at infinity. Let O be the center in finite space and $O_{/}$ the center at infinity (Fig. XXI). If the connector O $O_{/}$ is regarded as the ray m of the pencil O, its correspondent $m_{/}$ may be in finite space or the line at infinity.

If $m_{/}$ is in finite space the line at infinity is the ray $n_{/}$ of the pencil $O_{/}$ to which corresponds some ray n of the pencil O; this ray n is different from m and consequently does not pass through

$O_{\prime}$ but meets its correspondent $n_{\prime}$, which is the line at infinity, in a point at infinity different from $O_{\prime}$. Therefore the conic has two points at infinity, namely, the intersection of m and $m_{\prime}$ and the intersection of n and $n_{\prime}$, and consequently the conic is an hyperbola.

But if $m_{\prime}$ is the line at infinity, the curve is a parabola since this line $m_{\prime}^{\infty}$ is tangent at $O_{\prime}$ at infinity and all other intersections of corresponding rays are in finite space.

#34. Thus it follows from the above that the conic generated---

(a) by unequally directed pencils is always an hyperbola:

(b) by equally directed pencils with centers in finite space is an hyperbola, parabola, or ellipse, according as the number of pairs of parallel corresponding rays is two, one ,or zero:

(c) by equally directed pencils with both centers at infinity is an hyperbola:

(d) by equally directed pencils with one center at infinity and one in finite space is an hyperbola or a parabola according as the ray $m_{\prime}$ of the pencil with its vertex at infinity, which corresponds to the line m, joining the centers, thought of as belonging to the pencil with its vertex in finite space, is in finite space or is the line at infinity.

#35. Another method of investigating the nature of the conic generated by two equally directed projedtive pencils is by making use of the fact that

$$\tan(tx) \cdot \tan(s_{,}x_{,}) = \text{a constant},$$

given b y Steiner (1). In this formula t and s are two perpendicular rays of one pencil and $t_{,}$ and $s_{,}$, also perpendicular, are the rays of a second pencil corresponding to t and s respectively; and x and $x_{,}$ are any two corresponding rays; (tx) represents the angle measured from t to x and $(s_{,}x_{,})$ the angle from $s_{,}$ to $x_{,}$.

#36. In any two equally directed projective pencils there is always one pair of perpendicular rays as t and s whose correspondents, $t_{,}$ and $s_{,}$, are also perpendicular. These rays can be constructed as follows: let O and $O_{,}$ be two pencils with three rays $a_{,}$, $b_{,}$, $c_{,}$, of the pencil $O_{,}$, corresponding respedtively to a, b, c, of the pencil O. Rotate pencil $O_{,}$ until $b_{,}$ coincides with b, that is, until the pencils are in perspective with the line A C (Fig. XXII), the perspective cross section. Then describe a circle passing through O and $O_{,}$ having its center on A C. The points T and S, where this circle cuts A C are the points of intersection of the corresponding rays t, $t_{,}$ and s, $s_{,}$: also, s is perpendicular to t and s_1 to t_1 since their angle is inscribed in a semi-circle. If A C is the line at infinity, the pencils are equal and if the angle between s and t of one pencil is a right angle, so is the angle between s_1 and t_1.

37. Let the two pencils O and O_1 (Fig. XXIII) be such that the two perpendicular rays t and s correspond to two perpendicular rays t_1 and s_1; and let x, x_1 and y, y_1 be any two other pairs of corresponding rays. Since the pencils are projective their anharmonic ratios are equal, therefore (1)

$$\frac{\sin(sx)}{\sin(tx)} : \frac{\sin(sy)}{\sin(ty)} = \frac{\sin(s_1x_1)}{\sin(t_1x_1)} : \frac{\sin(s_1y_1)}{\sin(t_1y_1)} \quad (1)$$

But (sx) and (xt) are complementary angles;

then $\sin(sx) = \cos(xt) = \cos(tx)$.

Also $\sin(t_1x_1) = \cos(s_1x_1)$

and $\sin(sy) = \cos(ty)$

and $\sin(t_1y_1) = \cos(s_1y_1)$.

Then, substituting in (1),

$$\frac{\cos(tx)}{\sin(tx)} : \frac{\cos(ty)}{\sin(ty)} = \frac{\sin(s_1x_1)}{\cos(s_1x_1)} : \frac{\sin(s_1y_1)}{\cos(s_1y_1)}$$

or $\cot(tx) : \cot(ty) = \tan(s_1x_1) : \tan(s_1y_1)$.

Then $\tan(s_1x_1) \cdot \cot(ty) = \tan(s_1y_1) \cdot \cot(tx)$

(1) Salmon: Conic Sections, #56.

or $\tan(s_{,}x_{,}) \cdot \tan(tx) = \tan(s_{,}y_{,}) \cdot \tan(ty) =$ a constant.

This relation may be thought of as determining the projective relation between two pencils.

If the rays y and $y_{,}$ (Fig. XXIV) are thought of as revolving about their respective centers, the angle $(s_{,}y_{,})$ increases from zero to 90° while (ty) decreases from 90° to zero, when s and $s_{,}$ are considered the initial positions. Then there is one position where $(s_{,}y_{,}) = (ty)$. Likewise on the other side of t where t and $t_{,}$ are the initial positions, (tz) is increasing as $(s_{,}z_{,})$ decreases and therefore there is another position where $(tz) = (s_{,}z_{,})$. If the two pairs of rays in these positions are called g, $g_{,}$ and h, $h_{,}$ then from the above relation

$$\tan(s_{,}x_{,}) \cdot \tan(tx) = \tan(s_{,}g_{,}) \cdot \tan(tg) = \tan(s_{,}h_{,}) \cdot \tan(th) \cdot$$

But $\tan(s_{,}g_{,}) = \tan(tg)$ and $\tan(s_{,}h_{,}) = \tan(th)$.

Therefore $\tan(s_{,}x_{,}) \cdot \tan(tx) = \left[\tan(s_{,}g_{,})\right]^2 = \left[\tan(tg)\right]^2 = \left[\tan(s_{,}h_{,})\right]^2 = \left[\tan(th)\right]^2$

or $\tan(sx) \cdot \tan(t_{,}x_{,}) = \left[\tan(sg)\right]^2 = \left[\tan(t_{,}g_{,})\right]^2 = \left[\tan(sh)\right]^2 = \left[\tan(t_{,}h_{,})\right]^2$

and consequently $(tg) = (s_{,}g_{,}) = (ht) = (h_{,}s_{,})$

or $(t_{,}g_{,}) = (sg) = (h_{,}t_{,}) = (hs)$.

Steiner[1] calls this constant product the "power" of the projective

(1) Steiner: loc. cet. #13.

relation and the rays g, g_1 and h, h_1 the "power rays". Then since $(tg) = (ht)$, etc. it will be noticed that the angles (gh) and (g_1h_1) are bisected by the rays s,t and s_1, t_1 respectively, and also that the angles (gh) and (g_1h_1) are equal, since $(h_1t_1) = (hs)$ and $(t_1g_1) = (sg)$.

#38. It can be proved that the nature of the conic generated by the two projective pencils depends on the relative position of these power rays g, h and g_1, h_1 [1]. The proof and the nature of this dependence is given in the following paragraphs.

In this proof it will be necessary to make use of the relation[2] $\tan(sx) \cdot \tan(sy) = \left[\tan(sg)\right]^2 = \left[\tan(hs)\right]^2$ which expresses the condition that x and y are harmonic conjugates with respect to g and h. If the four rays are harmonic, their anharmonic ration[3] is equal to -1. That is, $(ghxy) = -1$

or $$\frac{\sin(gx)}{\sin(hx)} : \frac{\sin(gy)}{\sin(hy)} = -1$$

or $$\frac{\sin(gx)}{\sin(hx)} = -\frac{\sin(gy)}{\sin(hy)} \qquad (1)$$

Since the ray s is the bisector of (gh) (#37)

$$(gs) = (sh) = -(hs) = -(sg)$$

Also $(gx) = (gs) + (sx)$, $(gy) = (gs) + (sy)$

(1) Steiner: loc. cet. #14.

(2) Steiner: loc. cet. #8, nr. 14.

(3) Cremona : loc.cet. #68.

$(hx) = (hs) + (sx)$, and $(hy) = (hs) + (sy)$.

Then, by substitution, and expansion and dividing, the left hand member of equation (1) by $\cos(gs) \cdot \cos(sx)$ and the right hand member by $\cos(gs) \cdot \cos(sy)$ and using the fact that $(hs) = (sg)$, equation (1.) reduces to $\dfrac{\tan(sx) - \tan(sg)}{\tan(sx) + \tan(sg)} = - \dfrac{\tan(sy) - \tan(sg)}{\tan(sy) + \tan(sg)}$

$$\text{or } \frac{\dfrac{\tan(sx)}{\tan(sg)} - 1}{\dfrac{\tan(sx)}{\tan(sg)} + 1} = - \frac{1 - \dfrac{\tan(sg)}{\tan(sy)}}{1 + \dfrac{\tan(sg)}{\tan(sy)}} = \frac{\dfrac{\tan(sg)}{\tan(sy)} - 1}{\dfrac{\tan(sg)}{\tan(sy)} + 1} .$$

This relation is of the form $\dfrac{\frac{A}{B} - 1}{\frac{A}{B} + 1} = \dfrac{\frac{B}{C} - 1}{\frac{B}{C} + 1}$

which reduces to $\dfrac{A - B}{A + B} = \dfrac{B - C}{B + C}$, which reduces to $B^2 = A\,C$.

Therefore $\tan(sx) \cdot \tan(sy) = [\tan(sg)]^2$.

Comparing this with the projective relation of #37, namely,

$\tan(sx) \cdot \tan(t_{,}x_{,}) = [\tan(sg)]^2$, it is readily seen that $(t_{,}x_{,}) = (sy)$.

#39. Now suppose that O and $O_{,}$ are two projective pencils with s, $s_{,}$ and t, $t_{,}$ two pairs of corresponding rays, and with the angles (st) and $(s_{,}t_{,})$ each equal to a right angle. Let g, $g_{,}$ and h, $h_{,}$ be the power rays (#37). Then let pencil $O_{,}$ be shifted without rotation until it is concentric with pencil O. Now if these concentric projective pencils are cut by any transversal whatever, it may be looked upon as the base of two superposed projective ranges in which the rays of the concentric pencils cut the transversal. But the power rays of the pencils do not necessarily pass through the power points of the ranges. But if this transversal U (Fig. XXV and XXVI) is drawn parallel to one of the bisectors of $(st_{,})$ and $(s_{,}t)$, the power rays will pass through the power points, for the following reasons.

#40. Since the transversal U is parallel to the bisector of $(st_{,})$ it makes equal angles with s and $t_{,}$; and since $(t_{,}x_{,})$ equals (sy) (#38), U makes equal angles with y and $x_{,}$ (Fig. XXV). But $x_{,}$ is the correspondent of x, and y is the harmonic conjugate of x with respect to the power rays g and h. Therefore if it is required to find the ray $x_{,}$ corresponding to x, it may be found by locating y, the harmonic conjugate of x with respect to g and h, and then drawing the ray $x_{,}$ so that it makes with U an angle equal to (yU). In this way the ray of one pencil corresponding to any given ray of the other may be found.

#41. By this means the vanishing points of the two ranges may be shown to be the middle points of the segments cut off on the transversal U by the power rays. Let H, G, $H_{,}$, $G_{,}$ be the points of intersection of the transversal U (Fig. XXVI) with the power rays h, g, $h_{,}$, and $g_{,}$ respectively. Suppose I be taken as the middle point of G H; then its harmonic conjugate with respect to G and H is the point at infinity (#12). Therefore the harmonic conjugate of the ray O I with respect to g and h is the line O J^{∞} parallel to U . But by #40 the ray $O_{,}J_{,}$, corresponding to O J^{∞}, makes with U an angle equal to the angle between O I, the harmonic conjugate of O J^{∞}, and U . Therefore if a ray is drawn making with U an angle equal to angle (O I U) this ray is $O_{,}J_{,}$. And since J^{∞} is the point at infinity of the one range, its correspondent $J_{,}$ is the vanishing point of the other range. In the same way I can be shown to be the vanishing point . Now since $(t_{,}h_{,}) = (sh)$ and U is parallel to a bisector of $(st_{,})$, U makes equal angles with h and $h_{,}$; also $(sg) = (t_{,}g_{,})$ and therefore U makes equal angles with g and $g_{,}$ and the segments G H and $G_{,}H_{,}$ are equal. Then, since angle $(O_{,}J_{,}U) = (OIU)$, and I was taken as the middle point of G H , $J_{,}$ is the middle point of $H_{,}G_{,}$ and $I H = H_{,}J_{,} = J_{,}G_{,} = G I$. Then, since I and $J_{,}$ are the vanishing points, H, $H_{,}$ and G, $G_{,}$, the points in which the power rays cut the transversal U , are the power points of the ranges;

and therefore the double rays of the pencils will go through the double points of the ranges.

Thus, by finding the relation between the number of double points and the relative positions of the power points of two super-posed ranges, the relation between the number of double rays and the relative positions of the power rays of two concentric pencils may be determined. From this relation, the dependence of the nature of the generated conic on these relative positions of the power rays may be determined.

#42. In order to find the relation between the number of double points and the relative positions of the power points, suppose the two super-posed ranges of Fig. XXVI are on the line U of Fig. XXVII. With $I J_1$ as the diameter describe a circle and at I or J_1 erect a perpendicular $I A$ equal in length to $I G$. Through A draw a line parallel to $I J_1$, cutting the circle in two points, M and N. From M and N drop perpendiculars to $I J_1$, meeting the diameter in x and y respectively. Then these points are the double points of the two ranges. For $I x \cdot J_1 x = (x M)^2 = (I A)^2 = (I G)^2$, therefore x belongs to each range and is self-corresponding because the product of its distances from I and J_1 equals the power of the projective relation. Likewise y is double point.

It is easily seen that if I G were equal to one half of I $J_{/}$, A M would be tangent to the circle and there would be but one of these double points. Or if I G were greater than one half of I $J_{/}$, there would be no double points, for in that case A M would not meet the circle at all. Therefore when I H or its equal $J_{/}H_{/}$ is greater than half of I $J_{/}$ or, in other words, these two segments overlap each other there are no double points; when either H and $H_{/}$ or G and $G_{/}$ coincide there is one double point; and when I H and $J_{/}$ $H_{/}$ do not overlap there are two double points.

#43. Therefore when, in the two pencils, the angle (gh) overlaps the angle $(h_{/}g_{/})$, the segments G H and $H_{/}$ $G_{/}$, of the two ranges, also overlap and as in that case there are no double points on the ranges, there are therefore no double rays of the pencils. Also, when either the rays h and $h_{/}$ or g and $g_{/}$ coincide the points H and $H_{/}$ or G and $G_{/}$ concide and there is but one double ray. And when the angles (gh) and $(h_{/}g_{/})$ do not overlap the segments G H and $H_{/}$ $C_{/}$ do not overlap and there are two double rays.

Therefore, if the power rays g, $g_{/}$ and h, $h_{/}$ of the two given pencils are constructed, the nature of the conic can be determined by the relative positions of these rays. That is, if, when the $O_{,}$ pencil is moved without rotation so that $O_{,}$ coincides with O, (gh) overlaps $(h_{,}g_{,})$ the conic is an ellipse, as in Fig. XXVIII. If they do not overlap, as in Fig. XXIX, the conic is an hyperbola. And if h and $h_{,}$ or g and $g_{,}$ coincide, as in Fig. XXX, the conic is a parabola.

Now if the pencil O is revolved about its center while the other pencil remains fixed and the projective relation is maintained, it is easily seen that in order that there may be a change from the case where $(g_{,}h_{,})$ and (gh) overlap to that where they do not, or vice versa, there must be some position of the rotating pencil for which, either h and $h_{,}$ or g and $g_{,}$ coincide. Then as the one pencil is revolved about its center the conic changes from an ellipse to a parabola and then to an hyperbola, or vice-versa.

#44. If the power of the projective relation is given, the two perpendicular rays can be chosen at random, and the two pairs of power rays g, $g_{,}$ and h, $h_{,}$ constructed by means of the tangents. For example, suppose the power is given equal to 16. This means that

$$\tan(tx) \cdot \tan(s_{,}x_{,}) = 16 = [\tan(tg)]^2 = [\tan(s_{,}g_{,})]^2$$

or $\tan(tg) = \tan(s_{,}g_{,}) = \pm 4$. Then draw any two pairs of perpendicular rays s and t of the pencil O (Fig. XXXI) and $s_{,}$ and $t_{,}$ of the pencil O_1, corresponding to s and t respectively. With any radius and O as a center describe a circle; also with the same radius and O_1 as center draw an equal circle. If the first circle cuts t in M and the second cuts s_1 in M_1, then measure off four times the length of the radius in both directions on the tangents at M and at M_1. Then the tangent of the angle$(G\ O\ M) = \frac{G\ M}{O\ M} = 4$, and so with each of the four angles formed. And since $\tan(tg) = \tan(s_1 g_1) = \pm 4$, the line O G is the power ray g; $O_1 G_1 = g_1$; O H $= h$; and $O_1 H_1 = h_1$. Then, by #43, the nature of the conic can be determined by the relative positions of (gh) and ($h_1 g_1$).

#45. But if, in the place of the power being known, any three pairs of corresponding rays are given, it will be necessary to construct, first the perpendicular rays s, t and s_1, t_1 and then the power rays g, g_1 and h, h_1. This can be done as follows; Suppose any three pairs of corresponding rays a, a_1; b, b_1; and c, c_1 of the pencils O and O_1 (Fig. XXXII) are given. Then the perpendicular rays s, t and s_1, t_1 can be constructed by #36. In this way the angles that these particular rays make with the given rays are determined and so they can be drawn in the given pencils.

After these rays s and s_1, t and t_1 are found, the power of the projective relation can be determined by drawing (Fig. XXXII) any circle about O as a center and an equal one about O_1 as a center, and finding the tangents of (tb) and ($s_1 b_1$), namely, M N and $M_1 N_1$ respectively, considering the radius as unity. Then since

$$\tan(tb) \cdot \tan(s_1 b_1) = [\tan(tg)]^2 = [\tan(s_1 g_1)]^2 = M N \cdot M_1 N_1 ,$$

the tangent of (tg) and ($s_1 g_1$) can be found by constructing the mean proportional M P (Fig. XXXIII) between the lines M N and $M_1 N_1$. Then $M P = \tan(tg) = \tan(s_1 g_1) = \tan(th) = \tan(s_1 h_1)$, since the radius in every case is unity. Then measure off M P on the tangent at M and an equal segment $M_1 P_1$ on the tangent at M_1. The lines O P and $O_1 P_1$ are the rays g and g_1 respectively. This method (#35 to #45) of determining the nature of the conic generated by two equally directed projective pencils involves much more construction work than that of finding the double rays directly, except when the perpendicular rays are given. But if the power only is known, two projective pencils may be built up by the above constructions. Then the kind of conic generated depends upon the relative positions of the pencils, since it has already been seen that the conic changes as the pencils are revolved about their centers. Therefore if any three rays of two pencils are given the nature of the conic may be determined either by constructing the double rays or by constructing the pdwer rays.

V. THE CONIC GENERATED BY A PENCIL AND A RANGE.

#46. In the above discussions the relation has been either between two ranges or between two pencils. So it would be of interest to know what would be generated if the forms were a pencil and a range. In regard to this Reye[1] says that "if a range of points U and a sheaf[2] of rays S which lie in the same plane are related projectively to each other, and through each point of U be drawn a straight line parallel to the corresponding ray of S , these will either intersect in one point or will envelop a parabola. That is, if the sheaf of rays S is cut by the infinitely distant line of the plane, an infinitely distant range of points is obtained which is projective to U. If this is not perspective to U it will generate with U a sheaf of rays of the second order which contains the infinitely distant straight line, and consequently envelops a parabola." The sheaf of rays of the second order spoken of is simply another way of speaking of the connectors of corresponding points as a series of rays which are tangent to the conic.

VI. DEGENERATE FORMS OF CONICS.

#47. It sometimes occurs, either from the relative positions of

(1) Reye: Geometry of Position, Part I, Lecture VII, #129.

(2) Reye uses the word "sheaf" as an equivalent to the word "pencil".

the generating ranges or pencils, or from the special character of their projective relation that the conic is not an ellipse, parabola, or hyperbola but a degenerate conic. T. A. Hirst[1] gives the following three degenerate forms of conics;

(1) "The line-pair, whose points fill two simple rows, and whose tangents constitute a double pencil terminited by the axes of those rows.

(2) "The point-pair, whose tangents fill two simple pencils, and whose points constitute a double row terminiated by the centers of those pencils.

(3) "The line-pair-point, whose tangents fill a double pencil and whose points form a double row situated on one of the rays of that pencil."

Professor Cayley[2] defines these forms as follows:

(1) The line-pair is considered as a point determined by two lines through the point.

(2) The point-pair is considered as a line terminated by two points on this line.

(3) When in the point-pair the two points coincide or in the line-pair the two lines coincide Cayley calls it a line-pair-point.

(1) "Proceedings of the London Mathematical Society" vol. 1-2, no.18,p. 166.

(2) "Collected Mathematical Papers", vol. VI, p. 202.

#48. Ordinarily when there are two projective geometric forms no two distinct elements have coincident correspondents but when the projective relation is of a special kind the correspondents of all the elèments, except one, of each form are coincident and the correspondent of that excepted one is indeterminate. These excepted ones are called the exceptional elements.

When the projective relation of two ranges is of this special character either a point-pair or a line-pair may be generated. Two such ranges, on two non-coincident bases, U and $U_{,}$ (Fig. XXXIV) with the exceptional points A and $B_{,}$ not coinciding, will generate a point-pair. For every point on U except A is the correspondent of $B_{,}$ and every point on $U_{,}$ except $B_{,}$ corresponds to A. But since the lines joining corresponding points are tangents to the curve it appears that all of the tangents to the conic must pass through the points A and $B_{,}$, therefore the conic has degenerated into the point-pair (A $B_{,}$).

#49. If now, without changing the position of the bases, or the character of the correspondence, the exceptional points A and $B_{,}$ are made to coincide (Fig. XXXV) then the point-pair is transformed into a line-pair. Because every point on U or $U_{,}$ may be looked upon as a contact point since an infinite number of connectors coincide with each base or each base may be considered as an infinite number of tangents. Therefore the line-pair $U, U_{,}$ is generated.

#50. When the correspondence is of an ordinary kind the degeneration of the conic depends on the relative positions of the points on the bases or of the bases themselves. When two corresponding points coincide at the intersection of the two bases, the conic degenerates into a point-pair. For when two corresponding points A and $A_{,}$ (Fig. XXXVI) coincide, the ranges are in perspective and all of the connectors pass through one point S, while any line through the intersection of the axes may be regarded as a connector of the points A and $A_{,}$ and therefore as a tangent to the conic. Thus the intersection $A\ A_{,}$ and the center of perspective S compose a point-pair, the two points of which cannot coincide unless the correspondence becomes special.

#51. From the above it is readily seen that, in changing from an ellipse to an hyperbola, or vice-versa, as the ranges move along their bases, thereby altering their relative positions, the conic degenerates into a point-pair. For suppose the ranges U and $U_{,}$ (Fig. XXXVII) have their intersection A and $B_{,}$, their vanishing points I and $J_{,}$, and their contact points B and $A_{,}$. Then, as has been proved in #20 when the contact line A_1 B lies between the intersection and the vanishing line I J_1 , the conic is an ellipse. Now if the range on U_1 is moved until A_1 coincides with A (Fig. XXXVIII) then the two ranges will be in perspective and the point-pair (A S) will be formed.

Next let the range on U be moved until B coincides with A_1 (Fig. XXXIX) then the conic becomes an hyperbola since the contact line lies on the opposite side of the intersection from the vanishing line (from #21). Similar results may be obtained by moving both ranges at the same time.

#52. When the bases of two projective ranges coincide, the conic may have any one of three forms, namely, a real point-pair, an imaginary point-pair, or a line-pair-point, according as the two double points of the superposed ranges are real, imaginary, or coincident. When they are real, as A and B_1 (Fig. XL) any line through A and B_1, looked upon as a connector of A and A_1 or B and B_1 may be considered as a tangent to the conic. Also, since the connector of any other two points coincides with the coincident bases U and U_1 , any point on U U_1 may be regarded as a contact point. Thus the point-pair (A B_1) is formed. When the double points are imaginary the point-pair becomes imaginary or a double line with no real terminations and no real tangents not coinciding with the line. But if $B_{,}$ coincides with A , the line-pair-point is generated (from #47).

#53. If the projective relation between two pencils is of a special kind, the conic may degenerate into either a line-pair or a point-pair just as is the case with the ranges. When two pencils, whose centers are O and $O_{,}$ (Fig. XLI) and whose exceptional elements are a and $b_{,}$ have such a relation that every ray of O intersects its correspondent on $b_{,}$ and every ray of $O_{,}$ intersects its correspondent on a, the conic degenerates into a line-pair. Since every point on a and also every point of $b_{,}$ may be regarded as a point of the conic, being the intersection of two corresponding rays . And since the points on these two lines are the only points of the conic, pencils with such a correspondence generate a line-pair (a $b_{,}$) .

#54. Suppose, now, the correspondence is not changed but the pencils are revolved about their centers, O and $O_{,}$ (Fig. XLII) until the exceptional rays a and $b_{,}$ are made to coincide; then the degenerate conic will transform itself from the line-pair (a $b_{,}$) to the point-pair (O $O_{,}$) . For now every ray of O except a will intersect its correspondent in O . In the same way every ray of $O_{,}$ except $b_{,}$

intersects its correspondent in $O_{/}$. Thus all of the points of the conic are merged into the two points O and $O_{/}$, except the double point a $b_{/}$ which has become indeterminate. Every ray through O and $O_{/}$ may be regarded as a tangent since its correspondent coincides with O $O_{/}$. Therefore the conic has degenerated into the point-pair (O $O_{/}$).

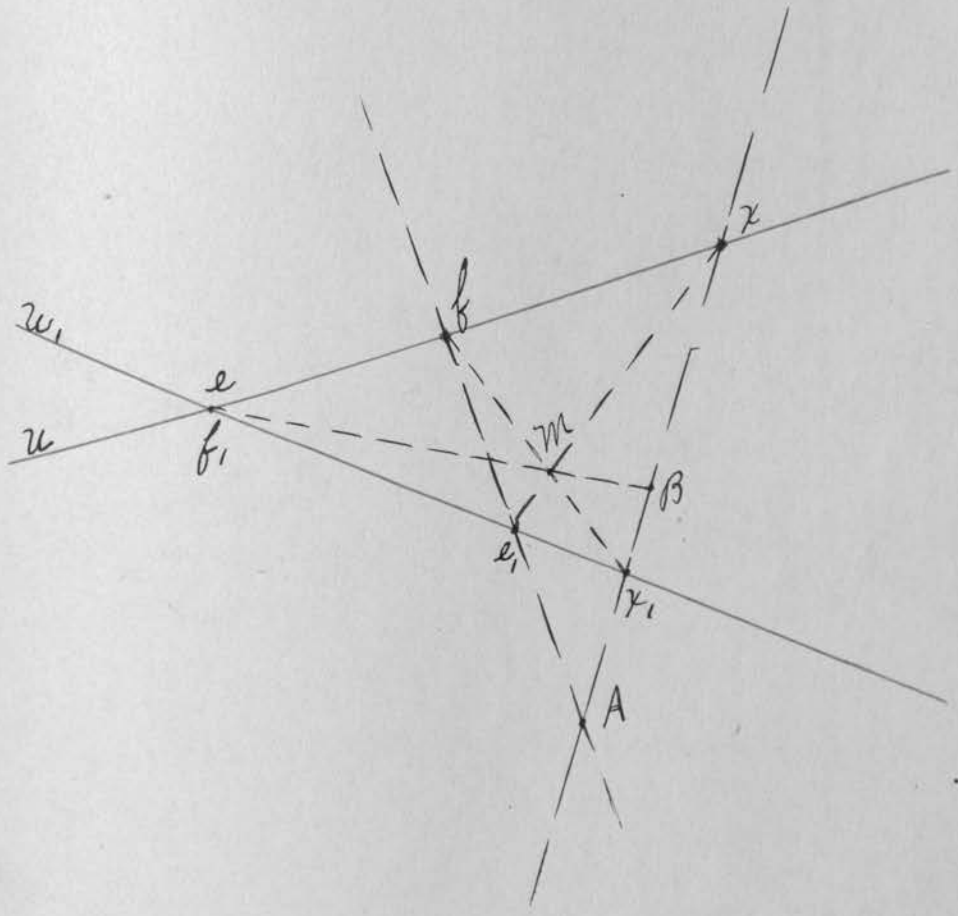
#55. When the correspondence of the two pencils is of an ordinary nature the degeneration of the conic depends on the relative position of the rays or centers. In these special positions there are two possibilities, first, when two corresponding rays coincide with the connector of the centers, and second, when the centers are coincident.

#56. In the first case a line-pair is generated (Fig. XLIII), for the pencils are then in perspective; and since by definition of perspective pencils the corresponding rays must meet on a straight line U, every point of U is a point of the conic. And as any point of the connector O $O_{/}$ may be considered the intersection of a and $a_{/}$, every point of O $O_{/}$ is also a point of the conic; therefore the line-pair is composed of U and O $O_{/}$.

By this result it is easily seen that any one of the proper forms of the conic generated by two projective pencils may be transformed into the line-pair by revolving the pencils about their centers until two corresponding rays shall coincide.

#57. If the centers of the two pendils of ordinary correspondence are made to coincide (Fig. XLIV) there may be three different forms, depending on the different projective relations. It is well known that when the centers are coicident there will be two and only two double rays a and b_1, also that these may be real, coincident, or imaginary, according to the nature of the correspondence of the pencils. When they are real, every point on either a or b_1 may be regarded as the intersection of the line with its correspondent. All other corresponding rays meet in the coincident centers O, O_1, the intersection of the double lines. Thus the form of the conic is the line-pair (a b_1). If the double rays coincide the two lines of the line-pair become coincident and the conic degenerates into a line-pair-point. But if the double lines are imaginary the form becomes an imaginary line-pair, to which the name point-ellipse is sometimes given, because all of its real points are coicident. Thus if one of the pencils is revolved about ~~about~~ their common center the conic transforms itself from a line-pair to a line-pair-point and then to an imaginary line-pair, since the double rays first being real become coincident and finally imaginary.

Figure I.



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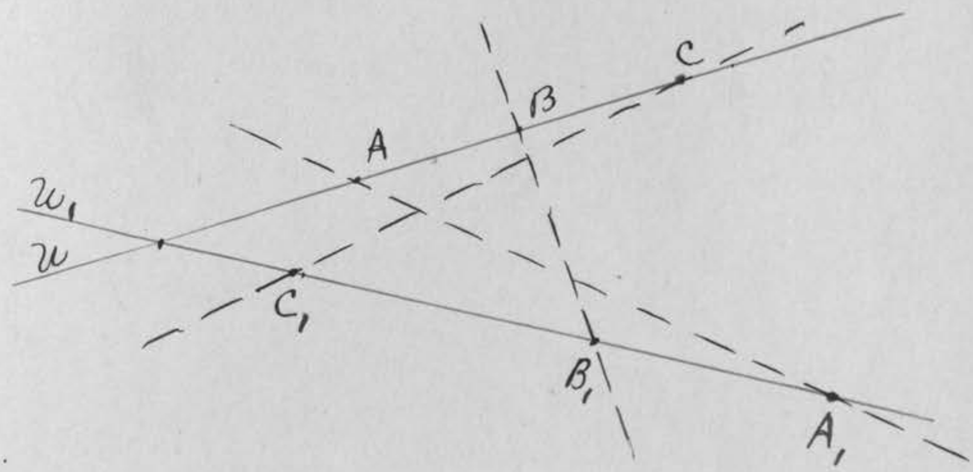

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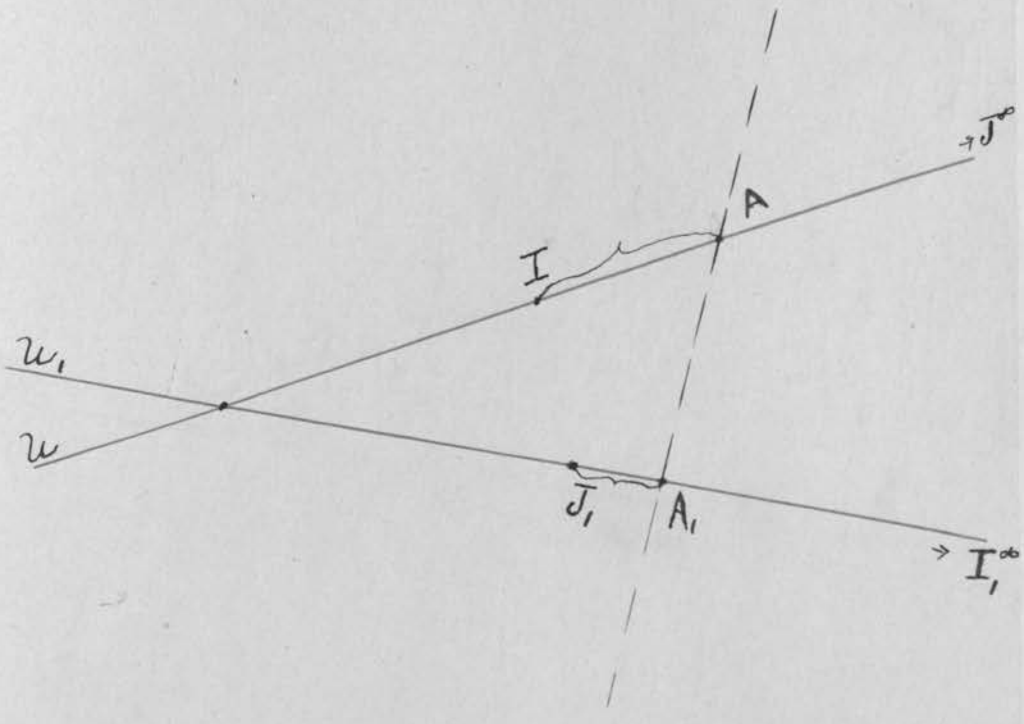

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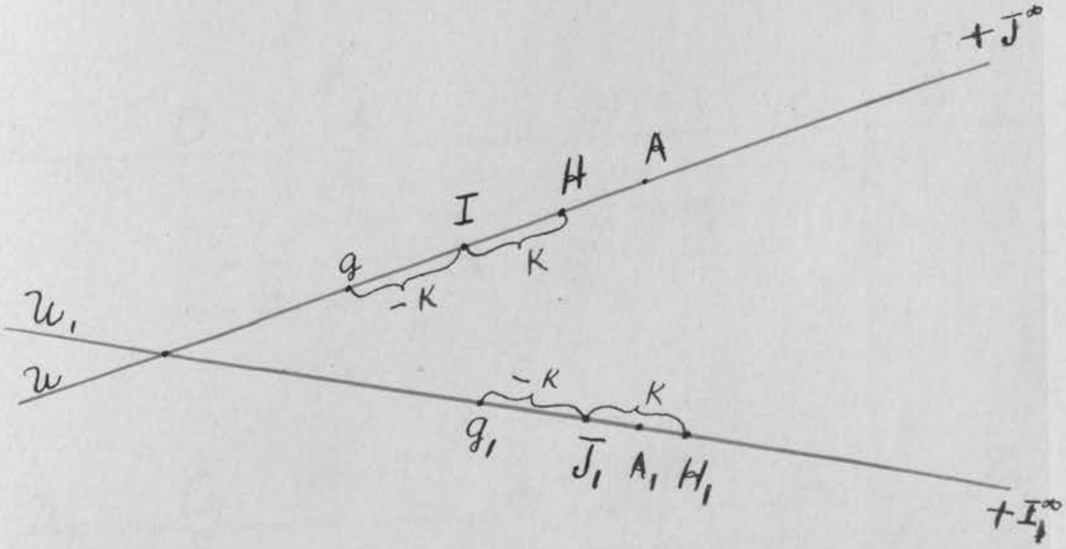

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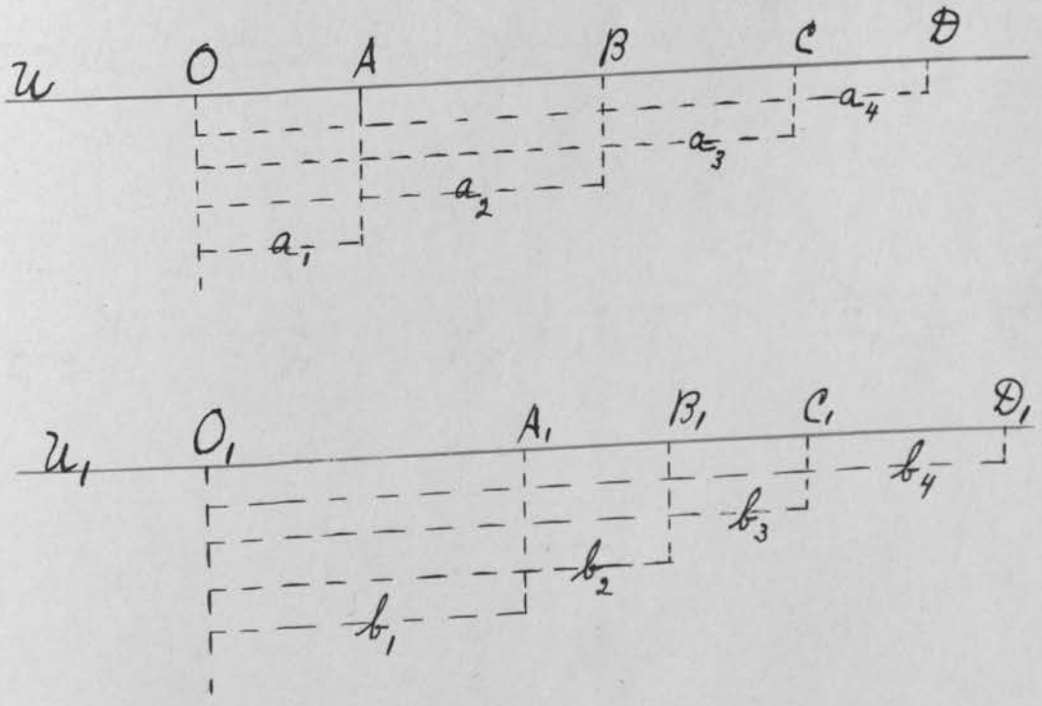

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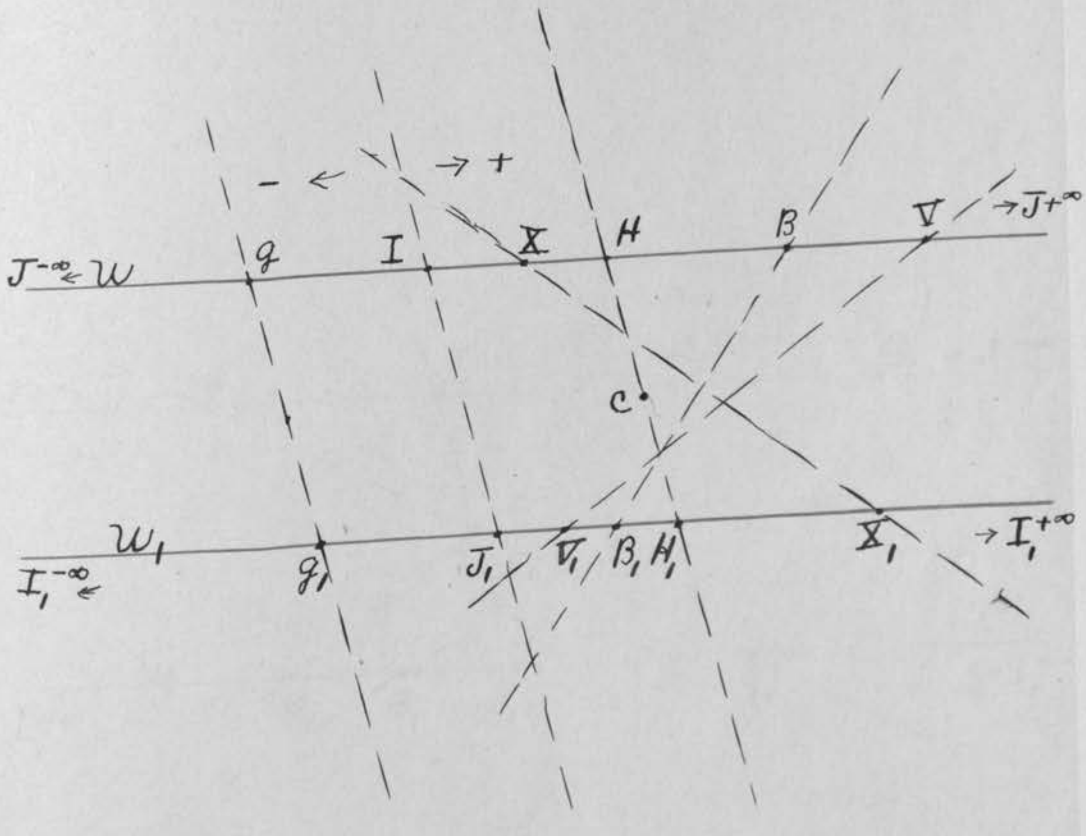

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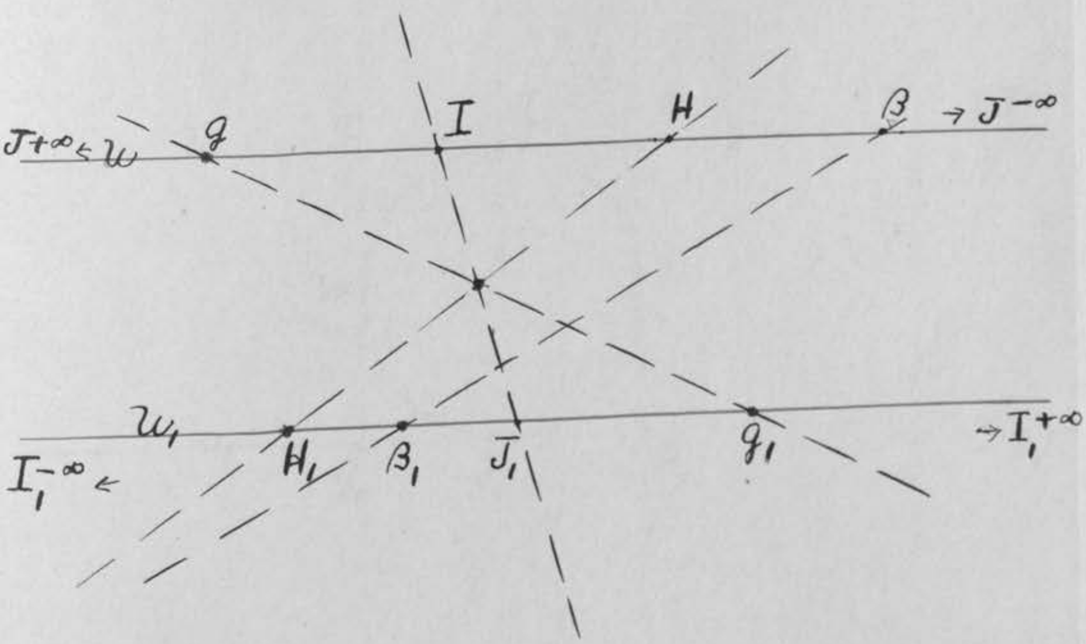
Figure VII.



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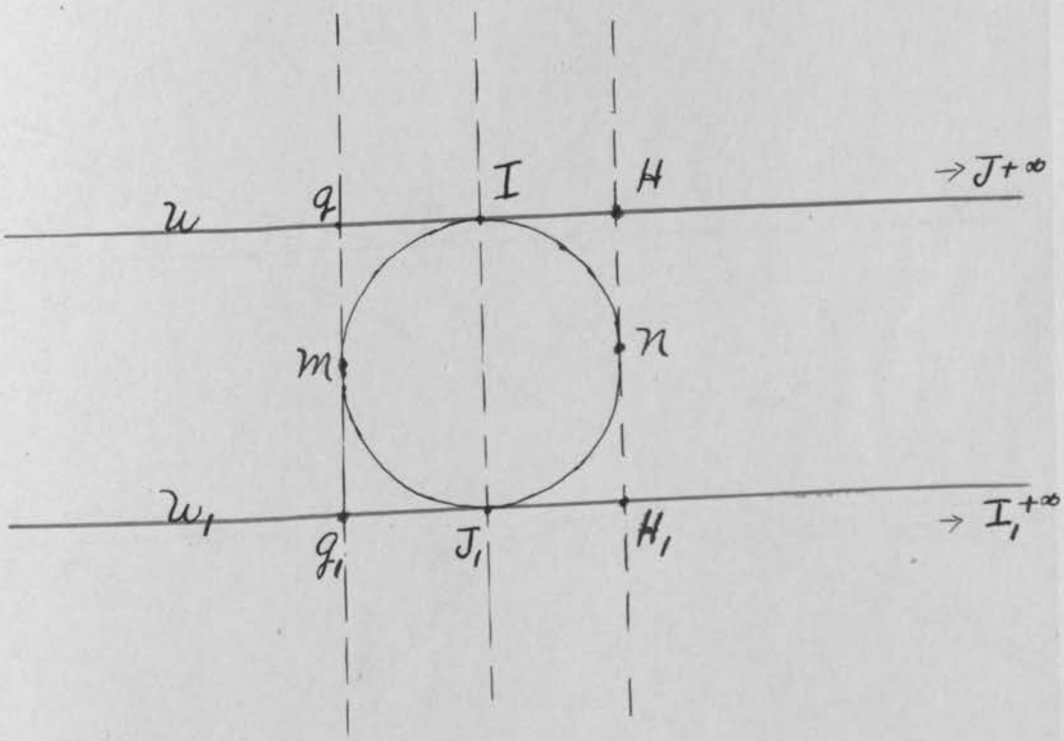

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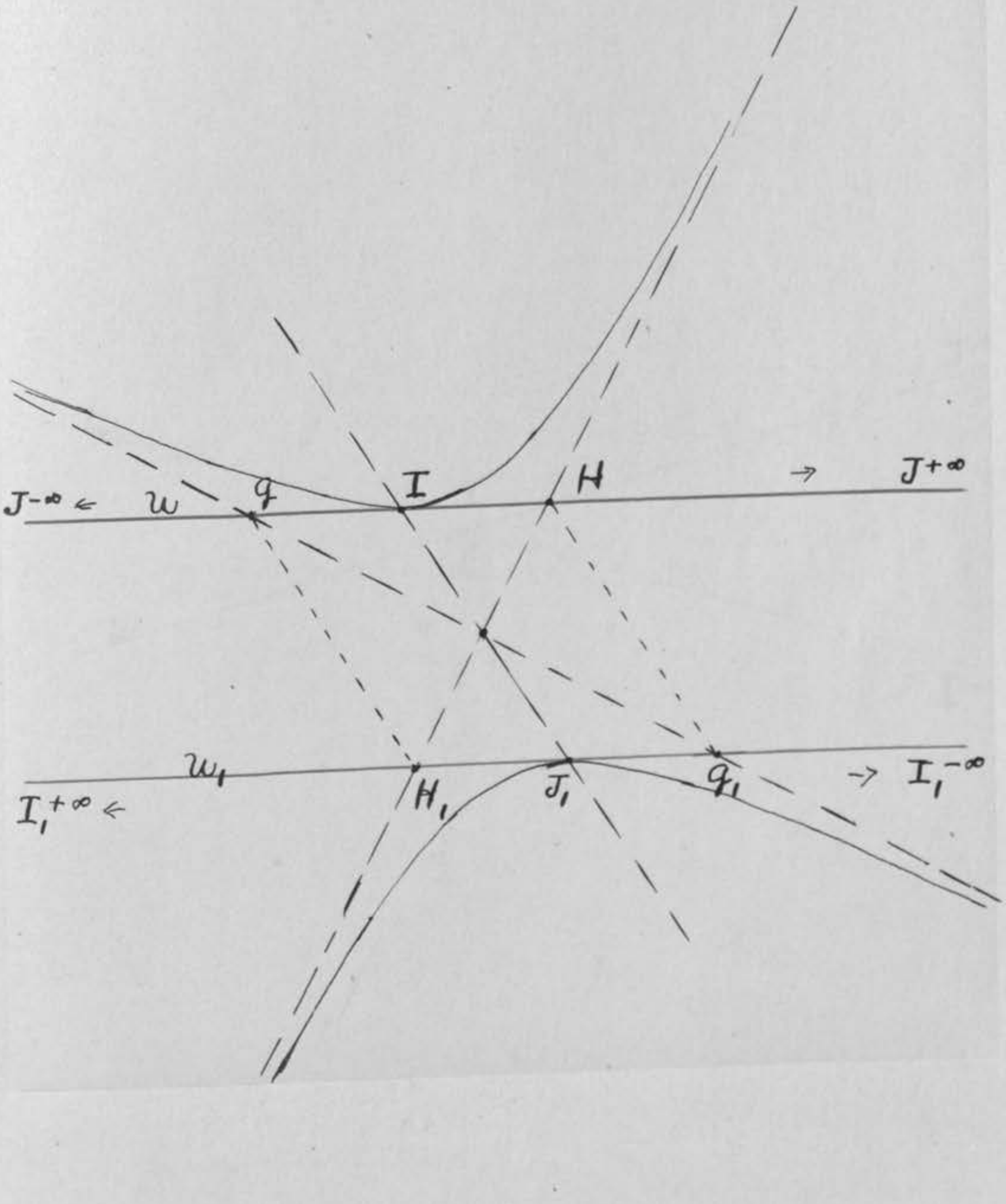

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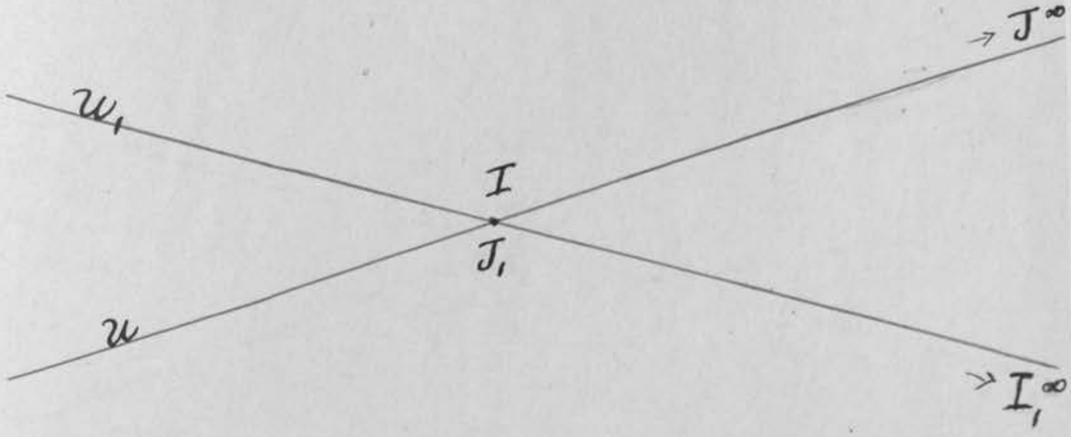

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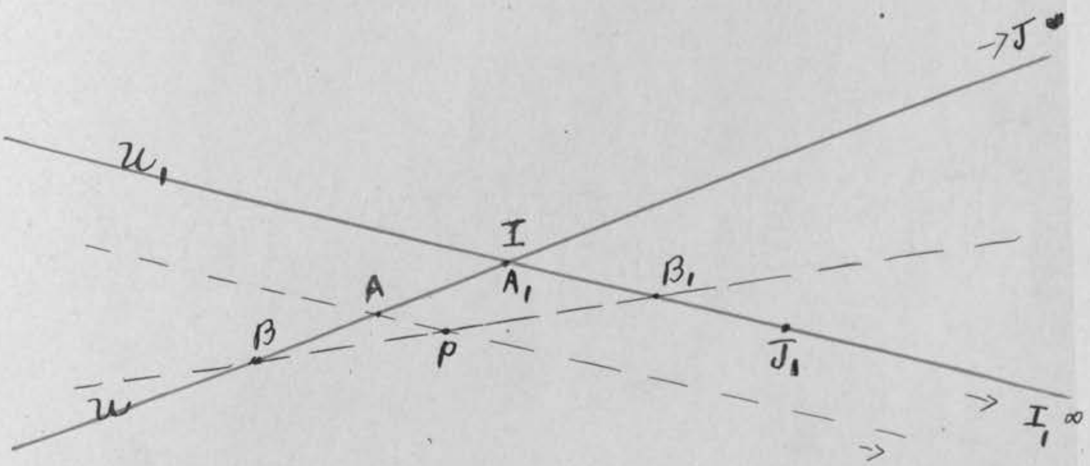

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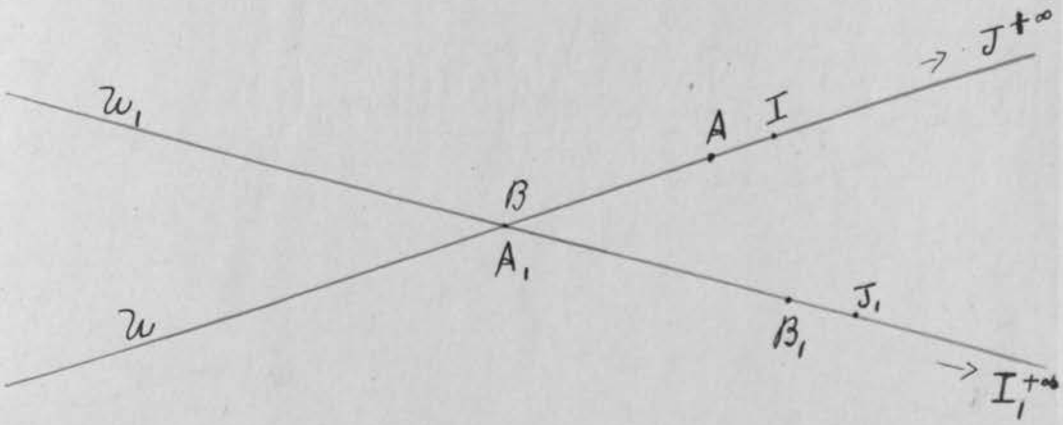

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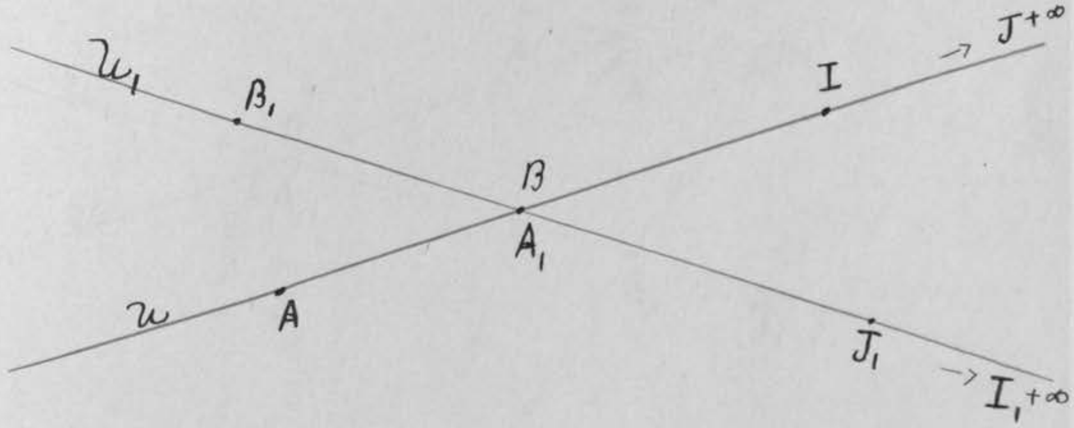

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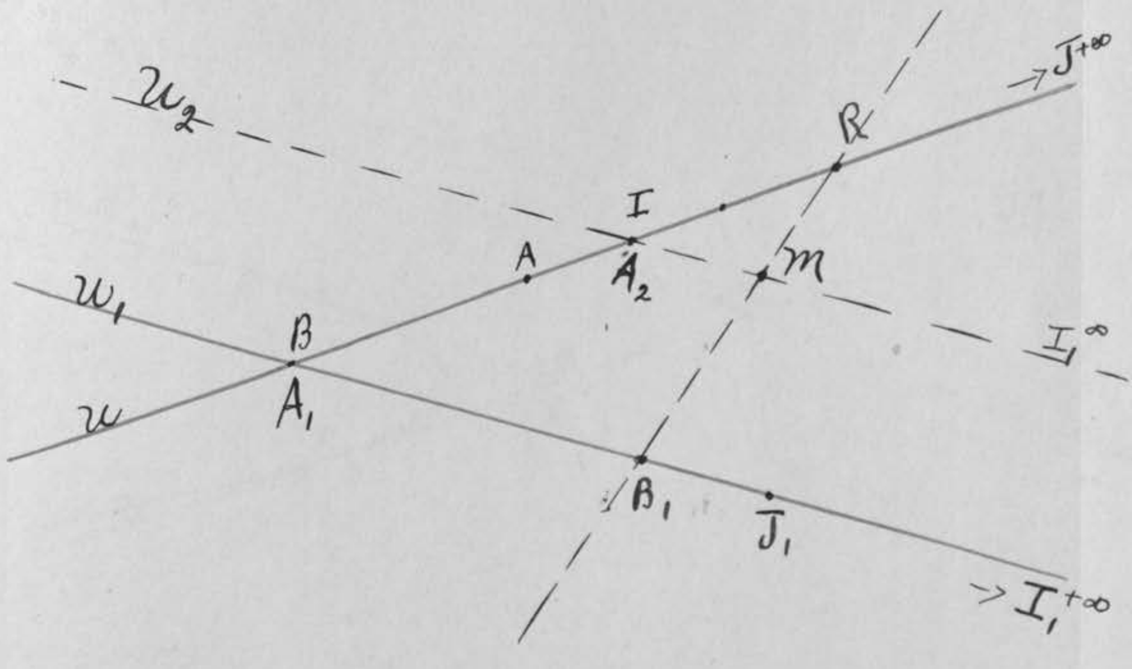

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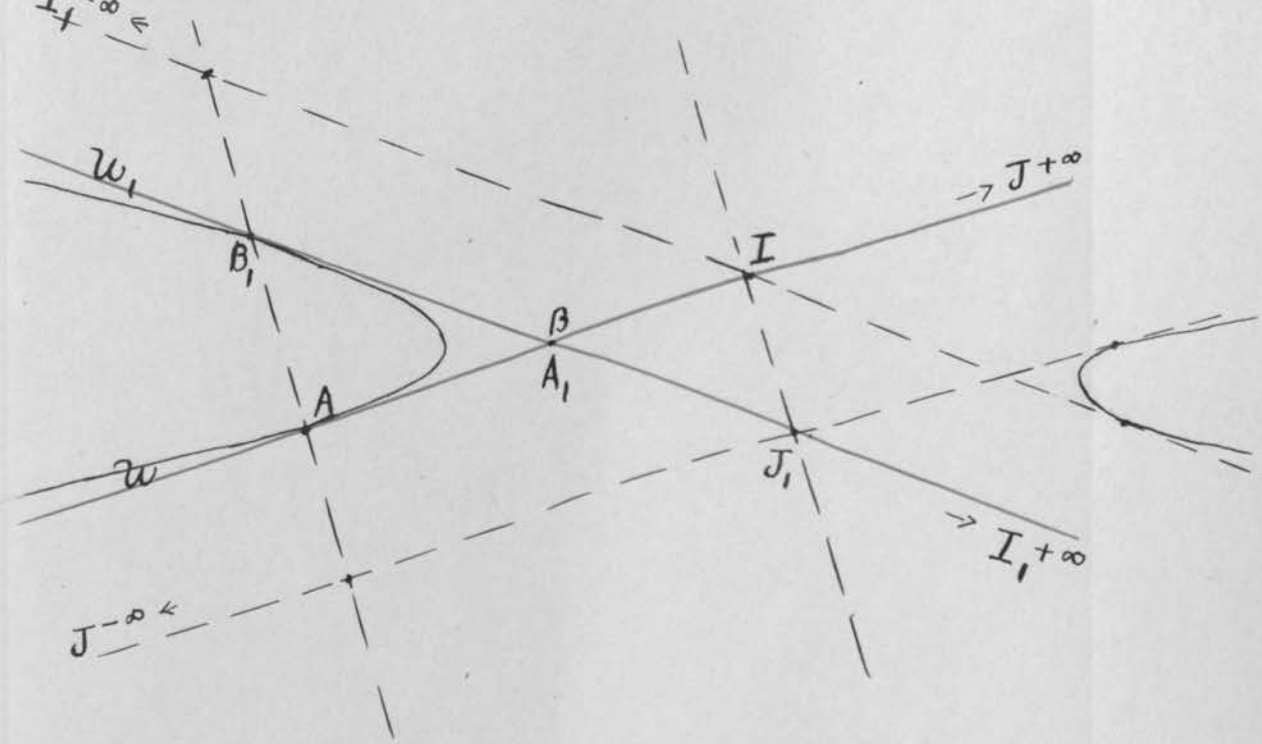

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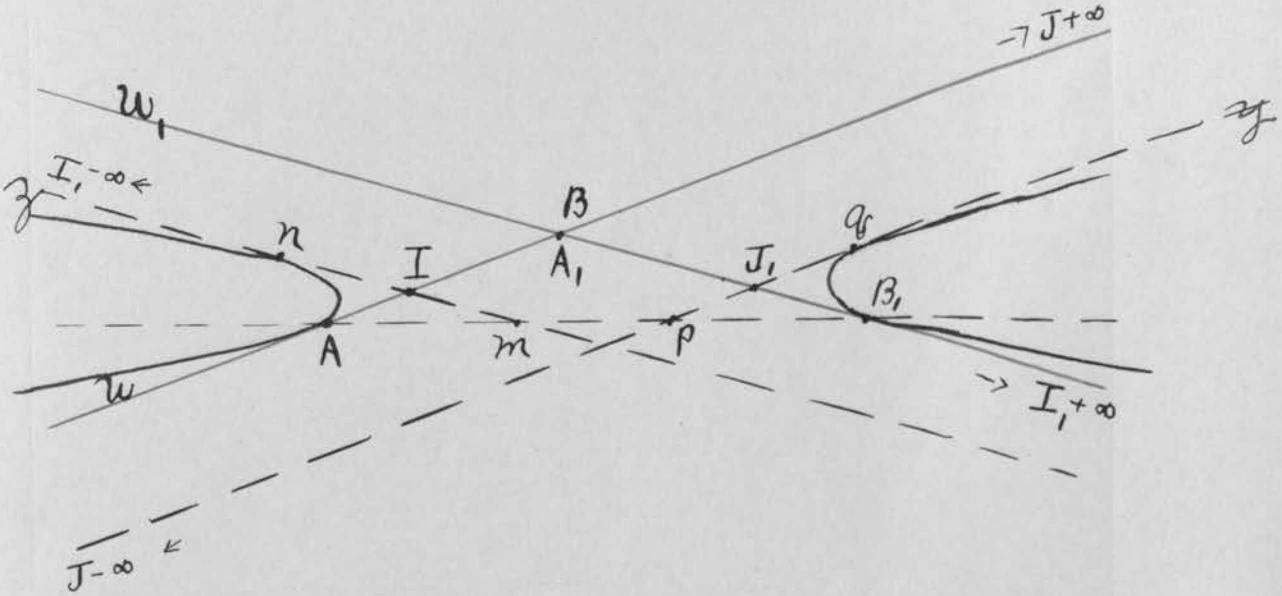

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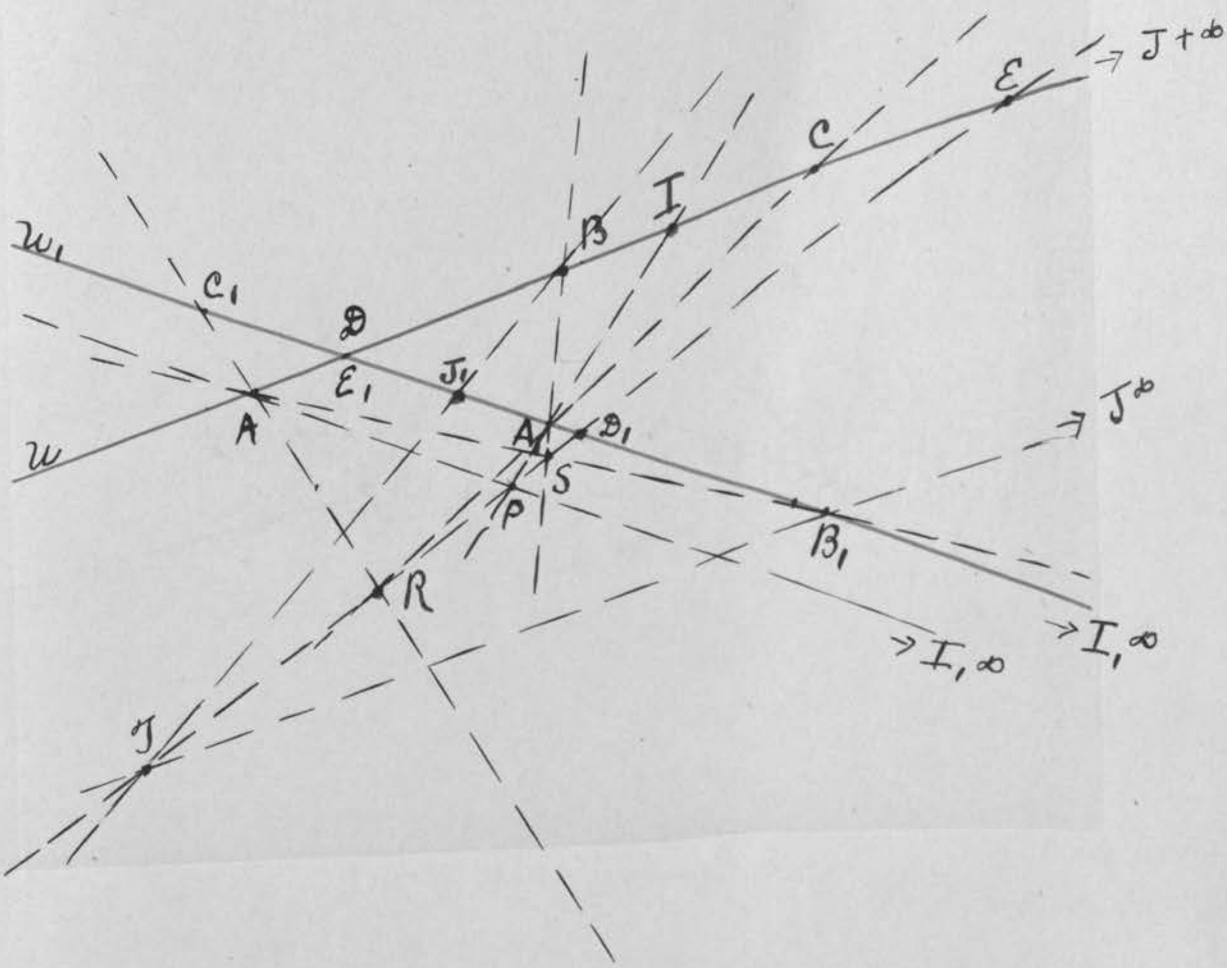

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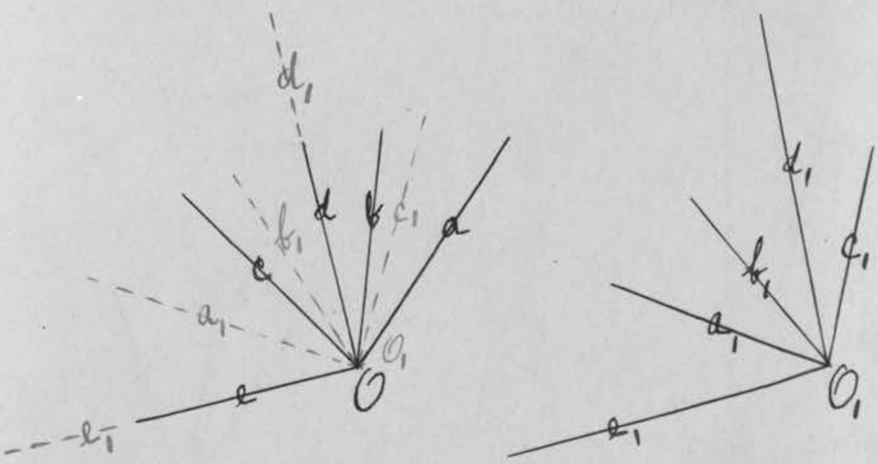
Figure XVIII



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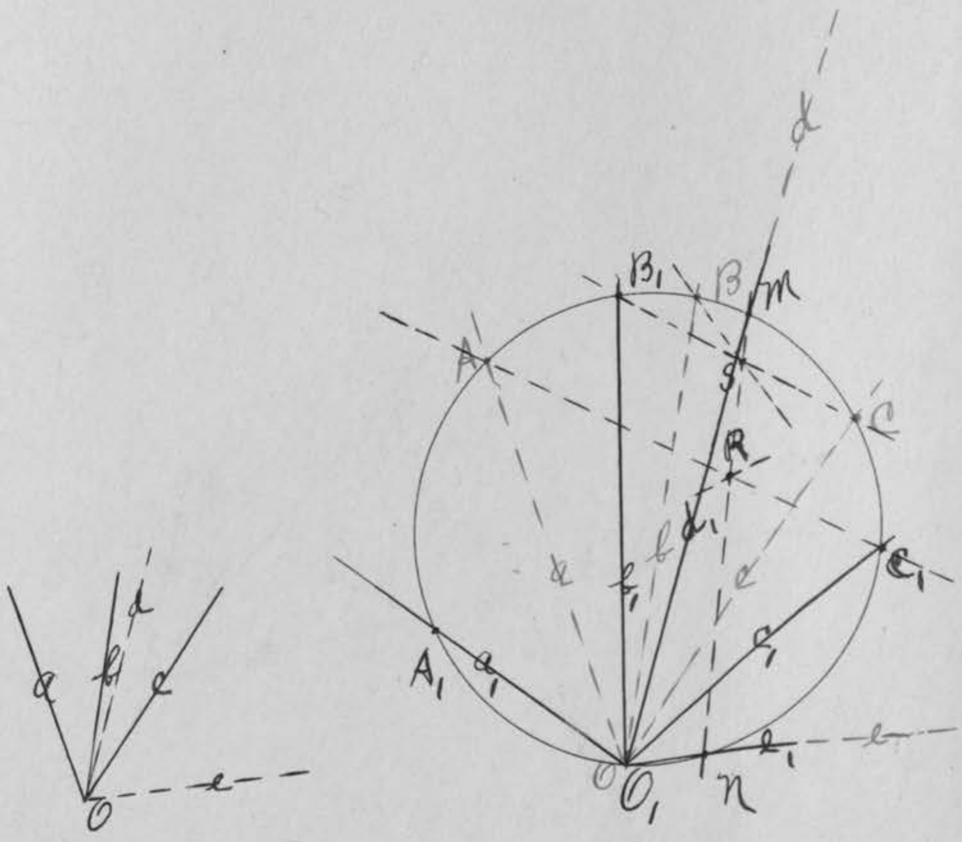

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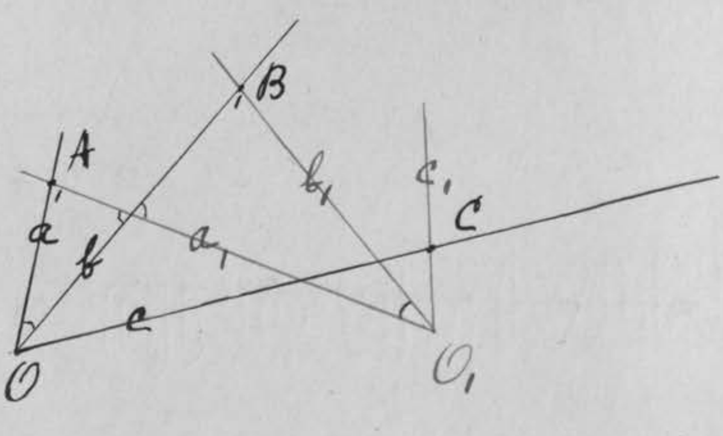

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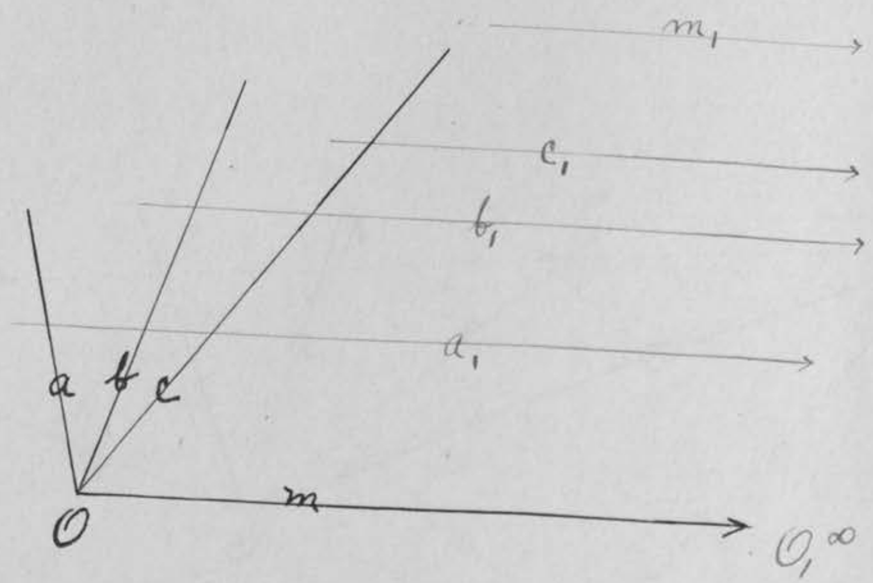

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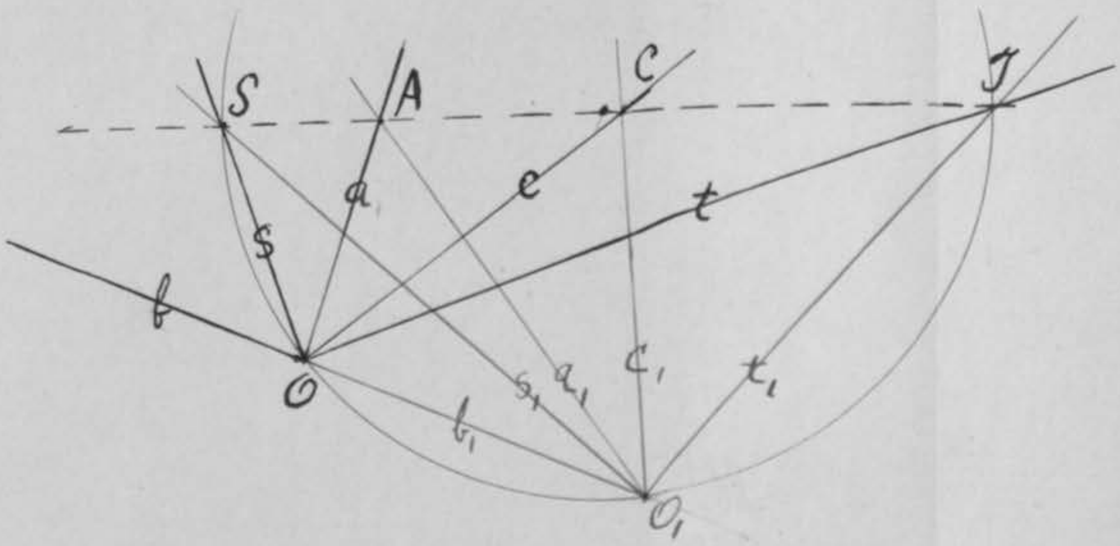

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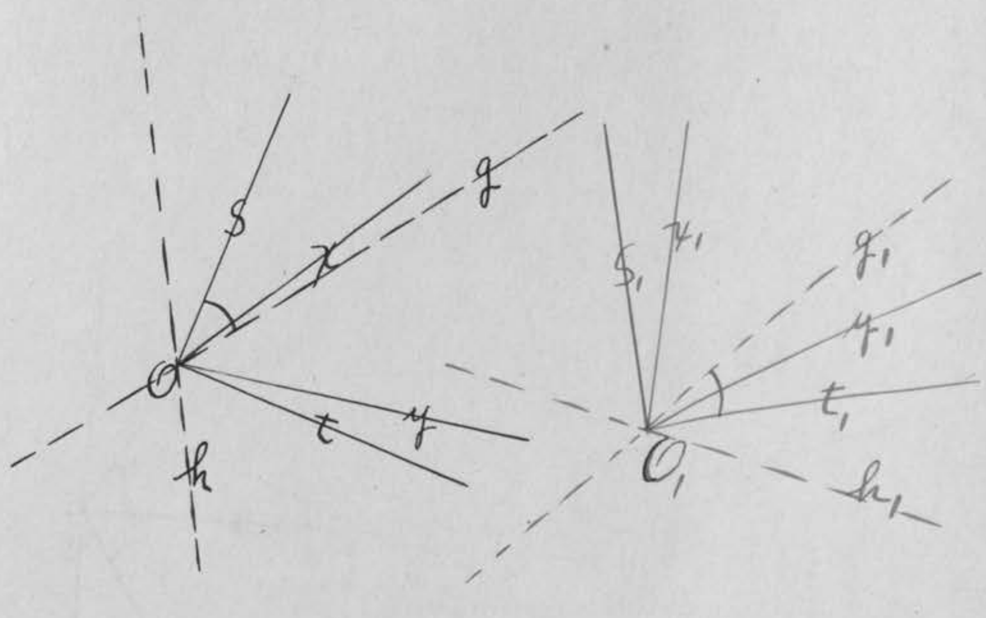

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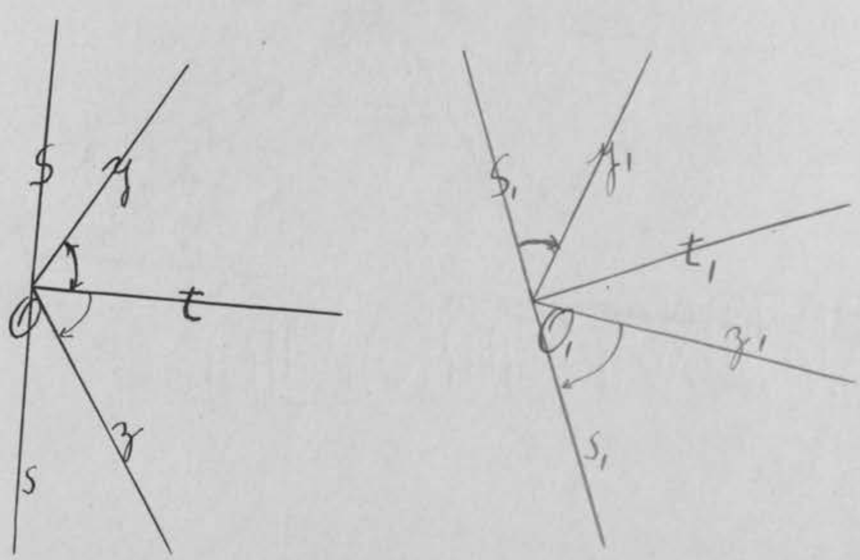

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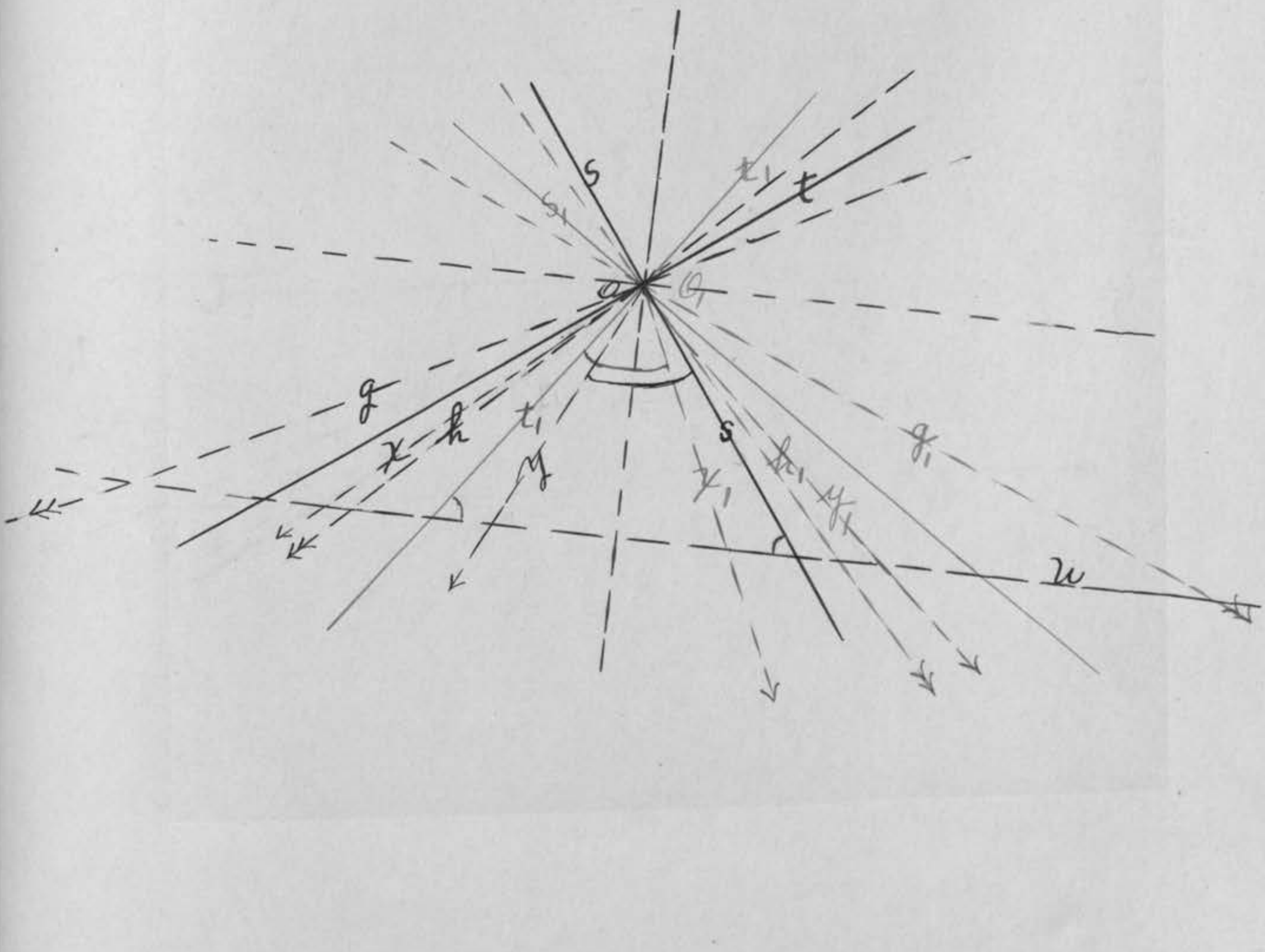

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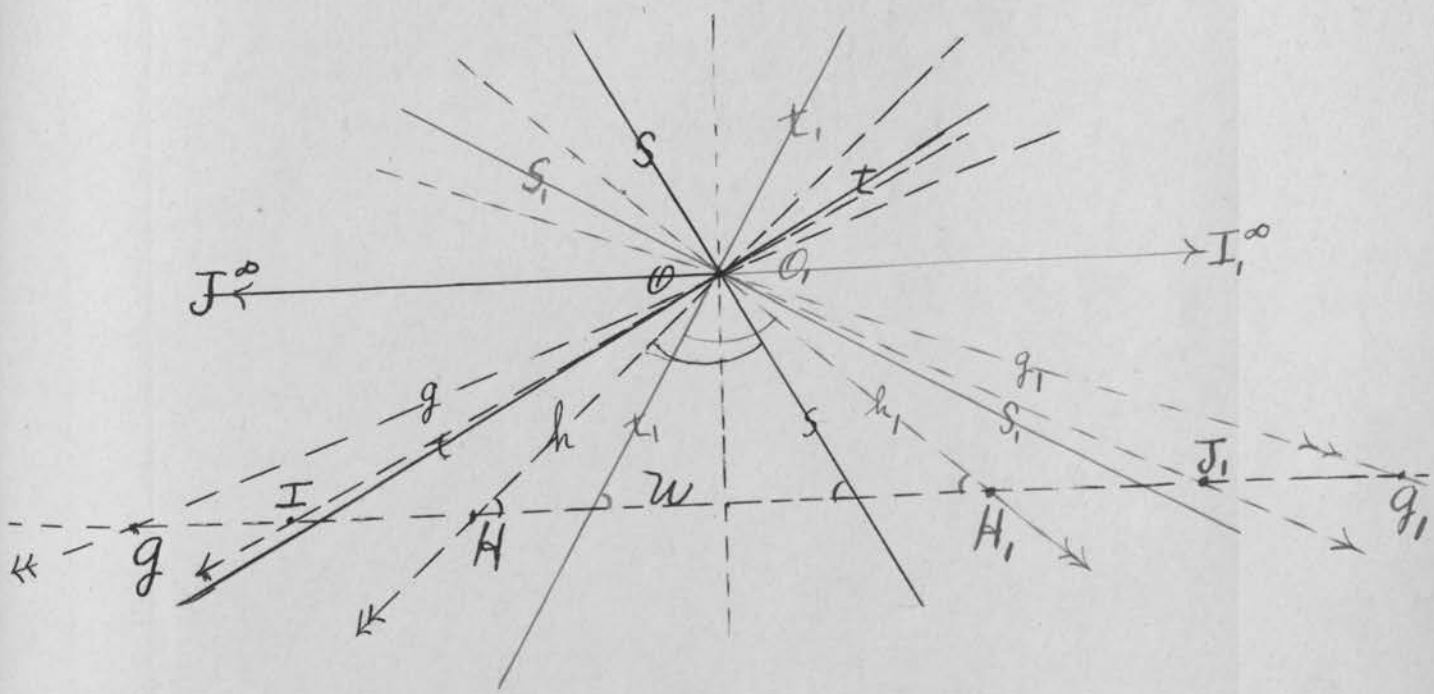
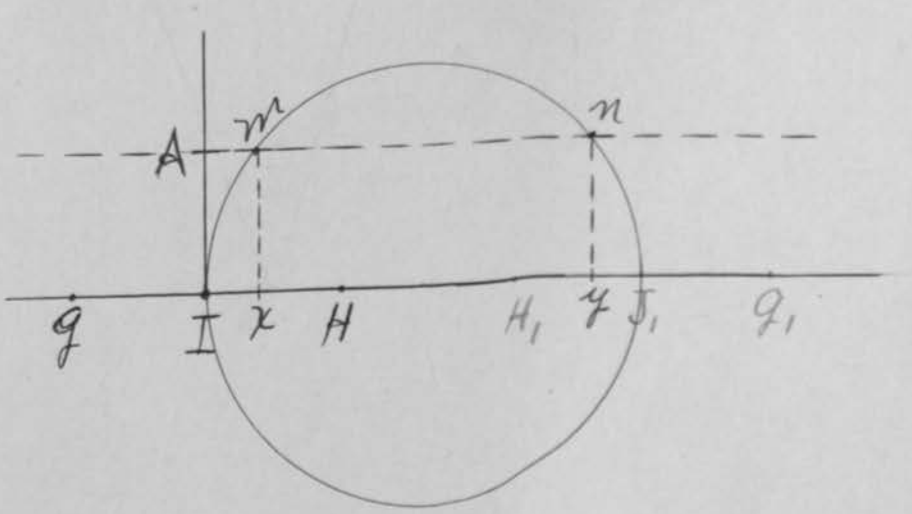


Figure $\boxed{\dot{X}XVII}$.



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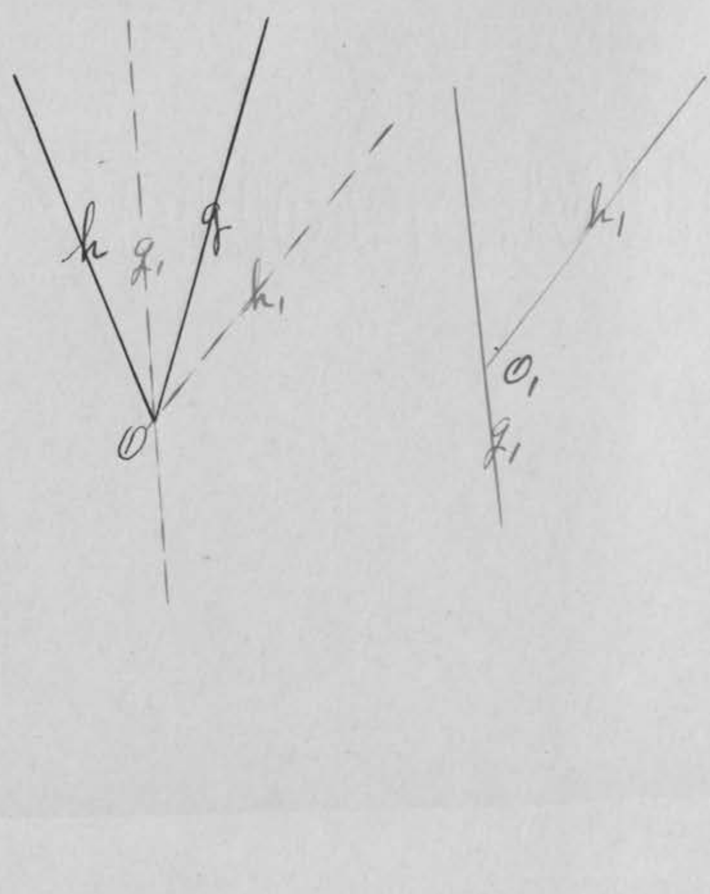

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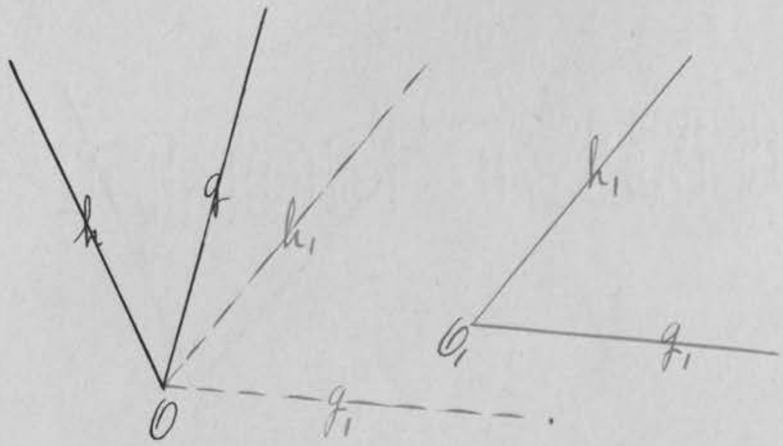

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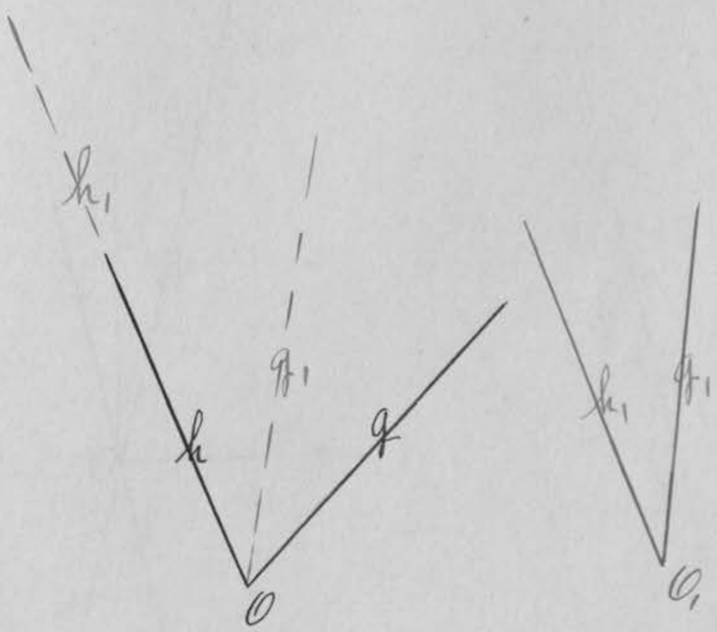

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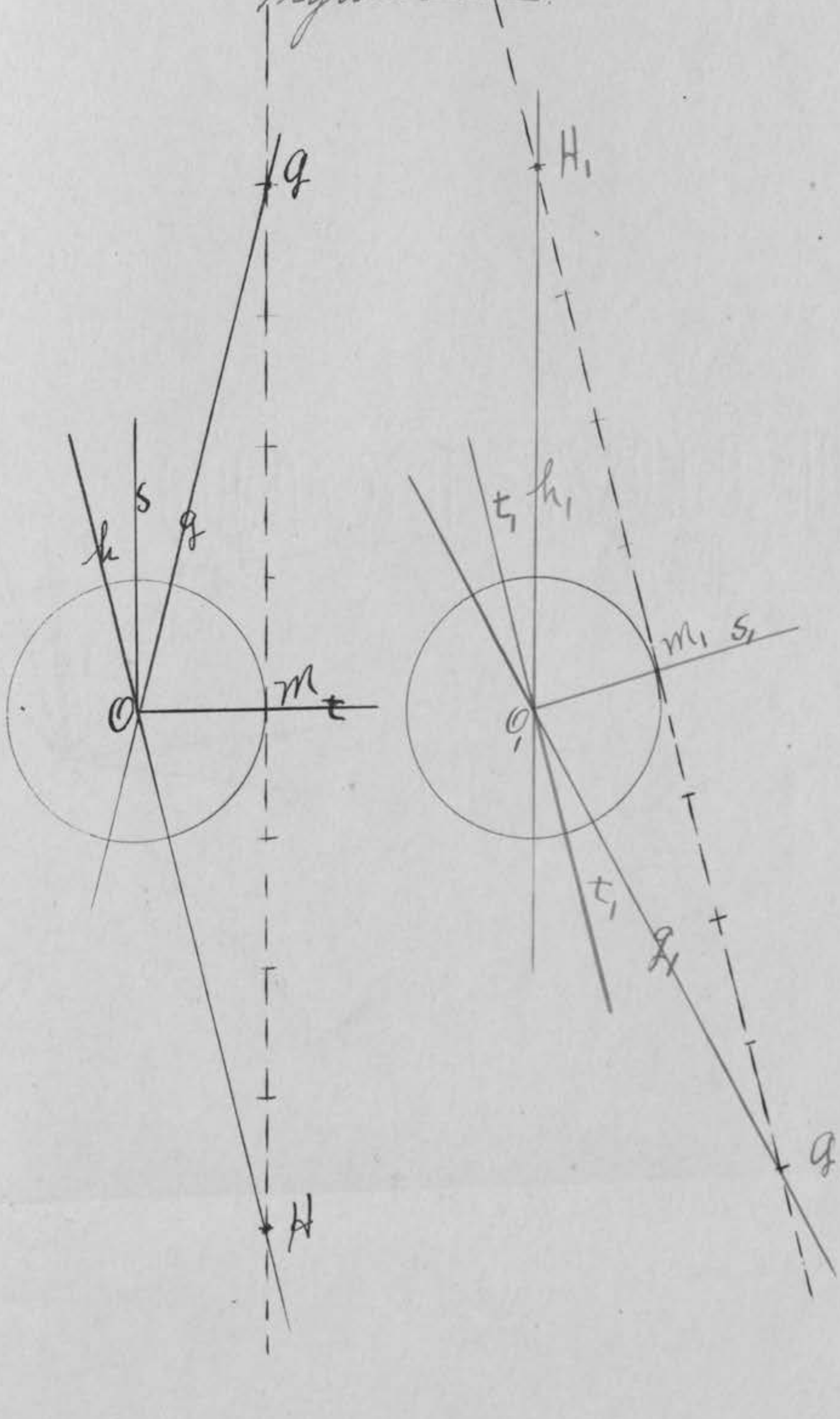

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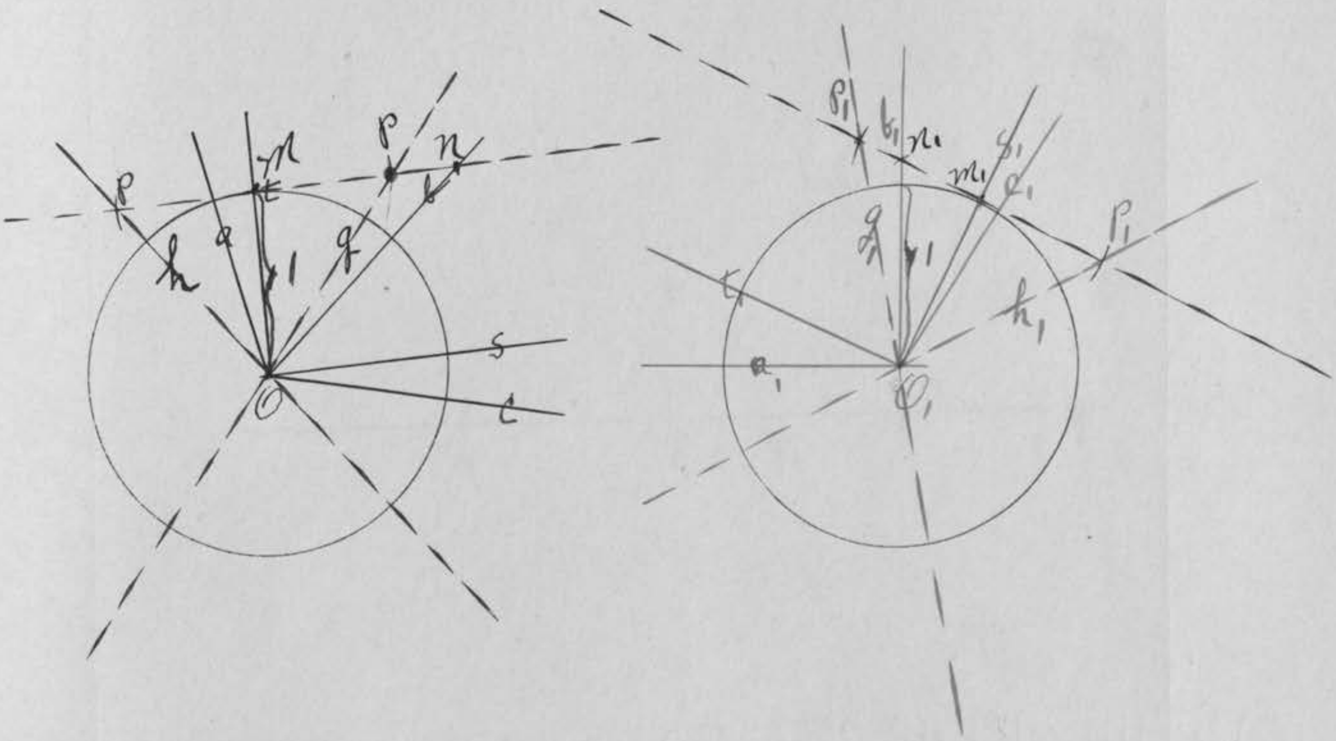

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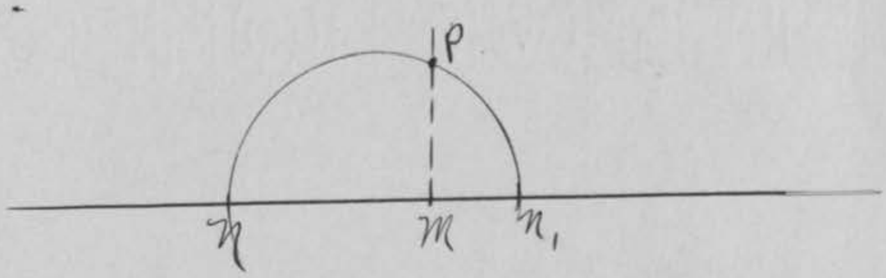

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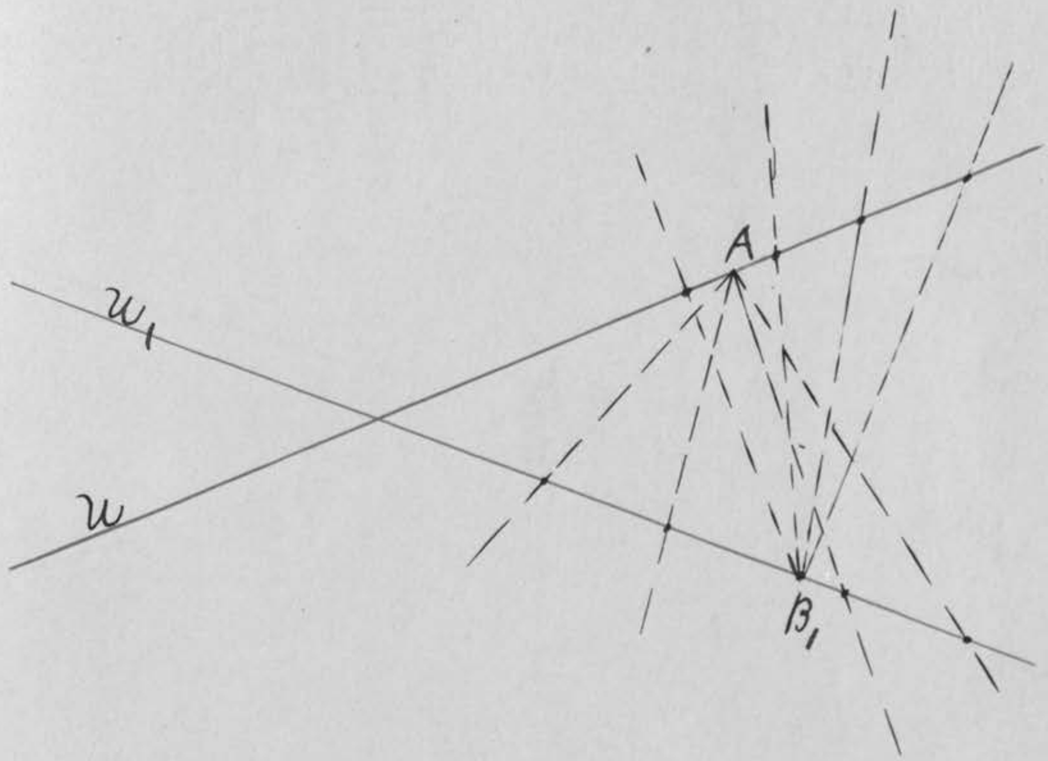

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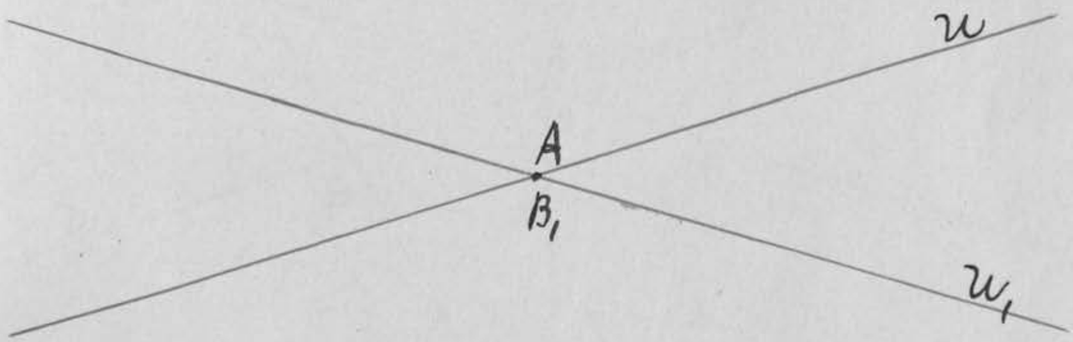

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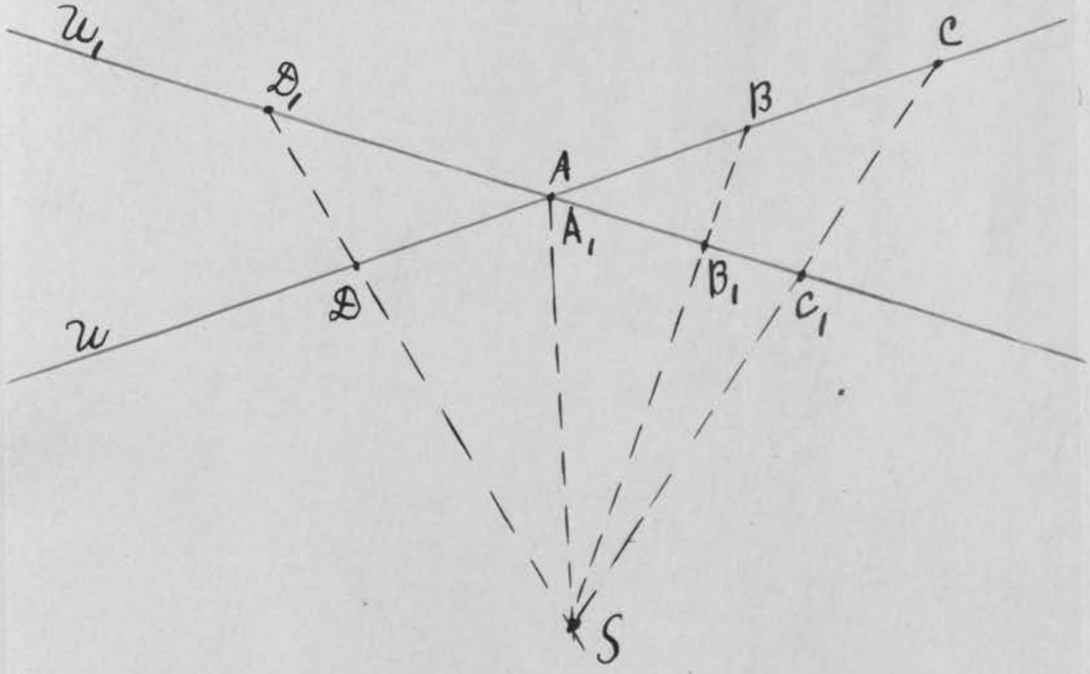

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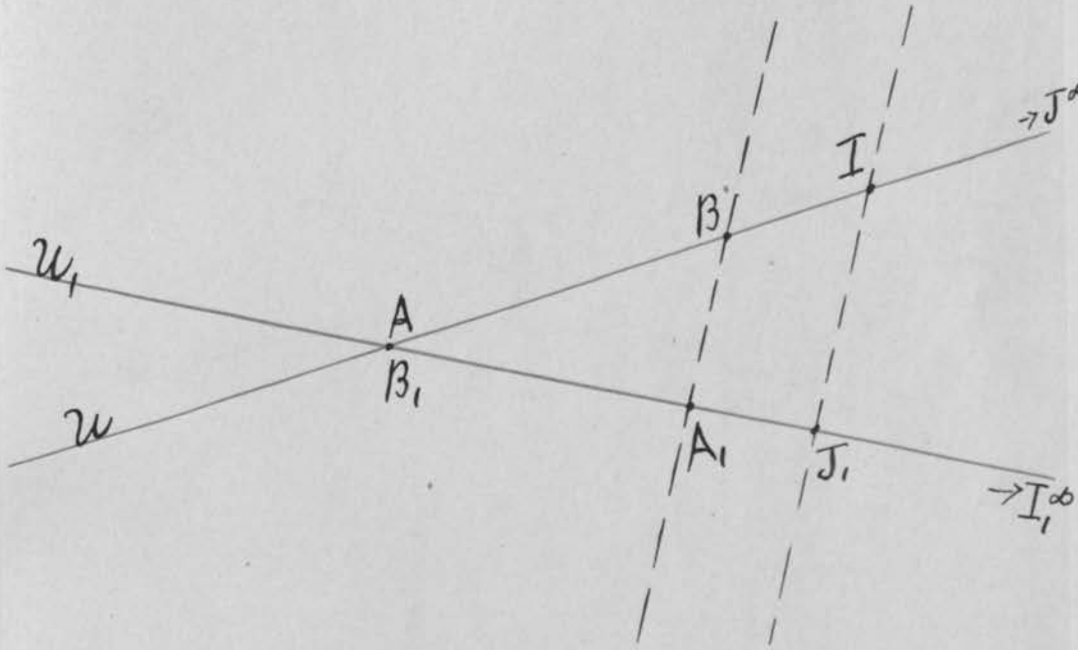

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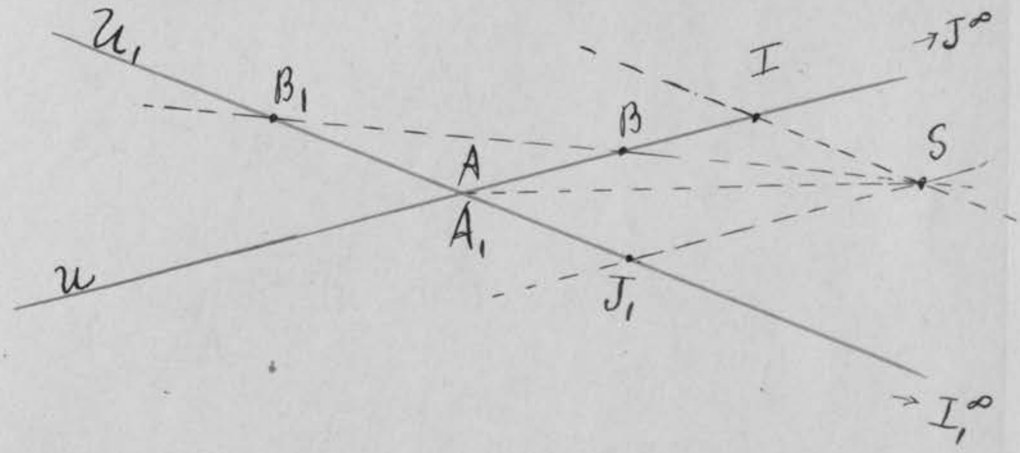


Figure $\underline{XXXIX}$.

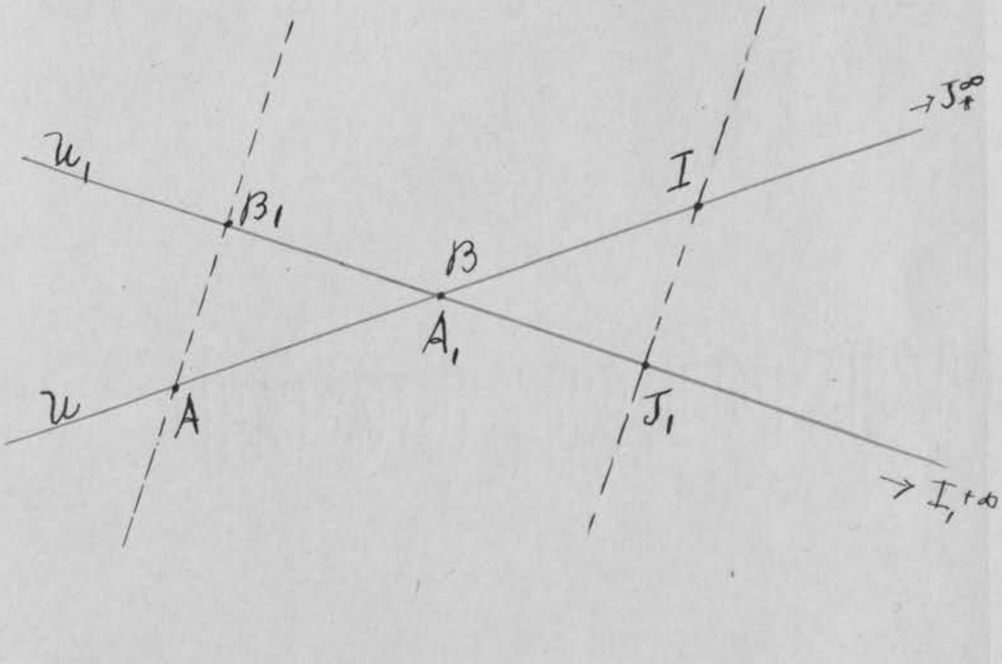


Figure XL.

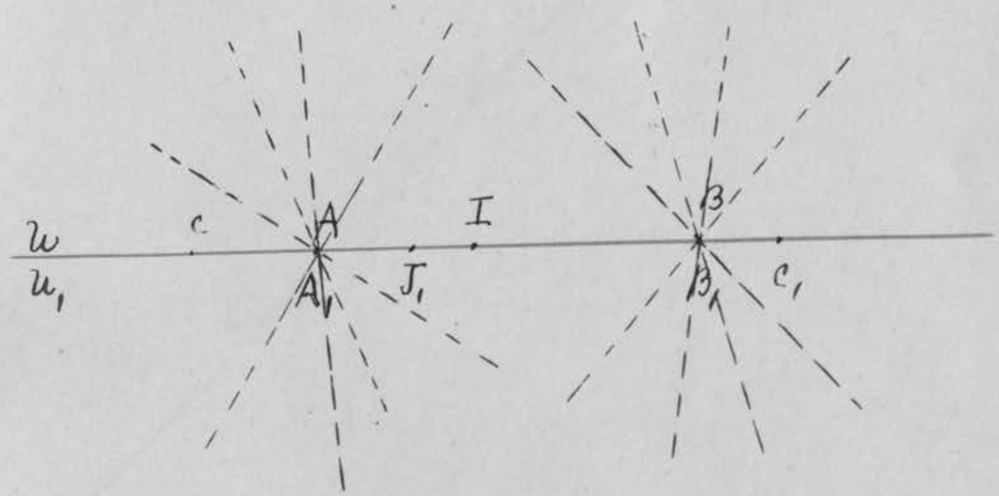


Figure XLI.

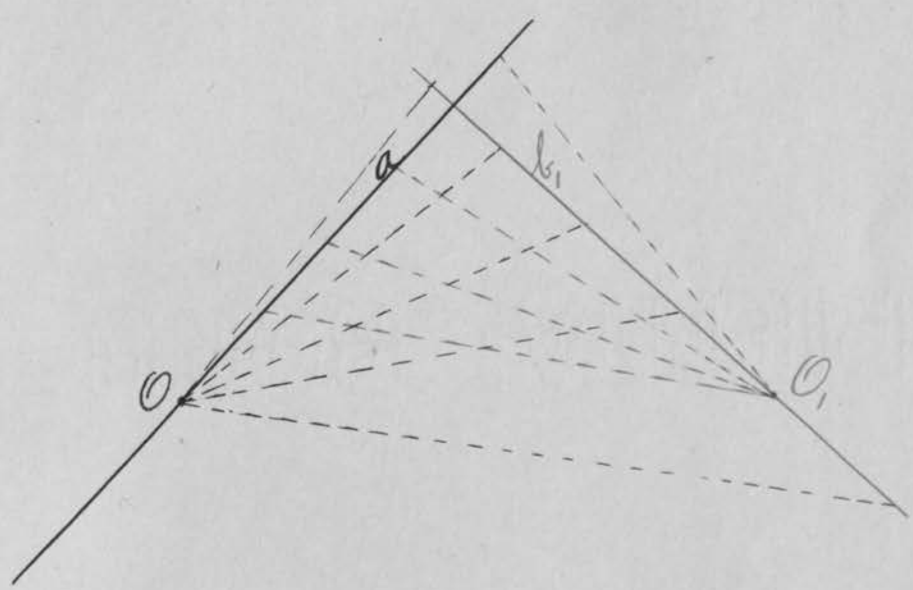


Figure XLII.

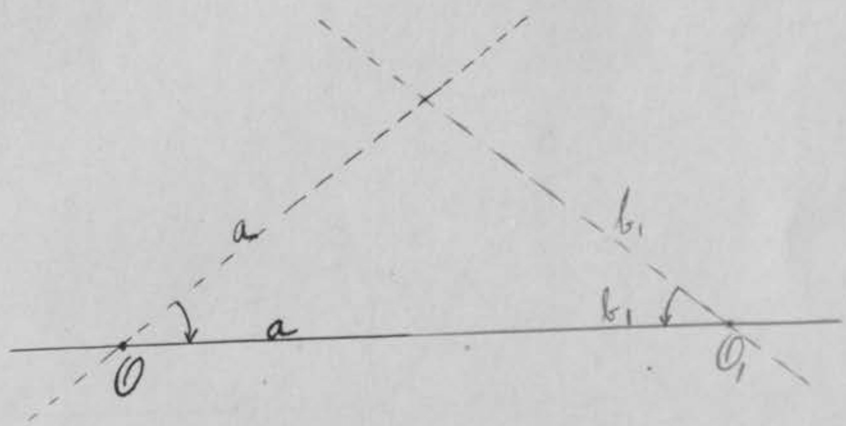


Figure XL, III.

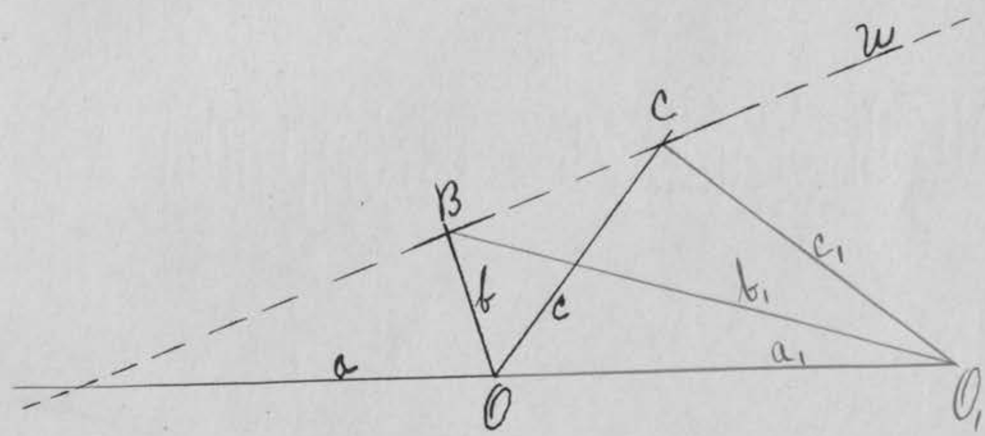
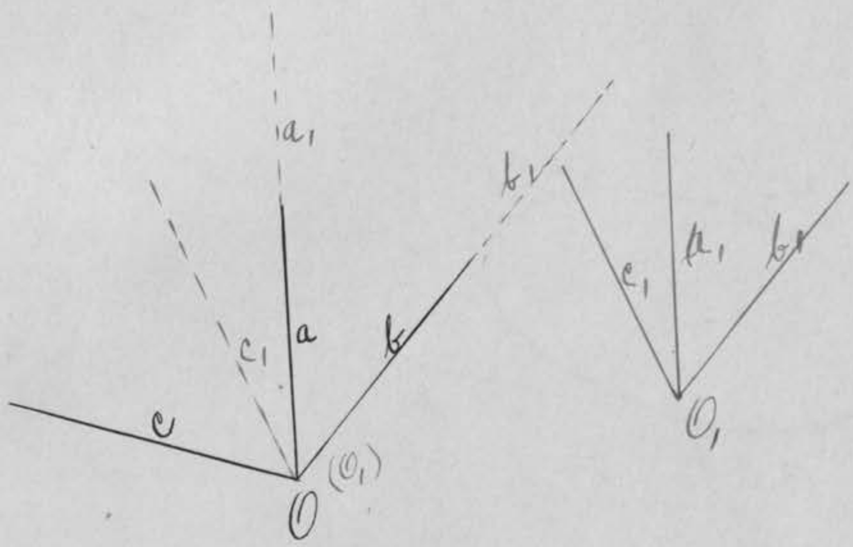
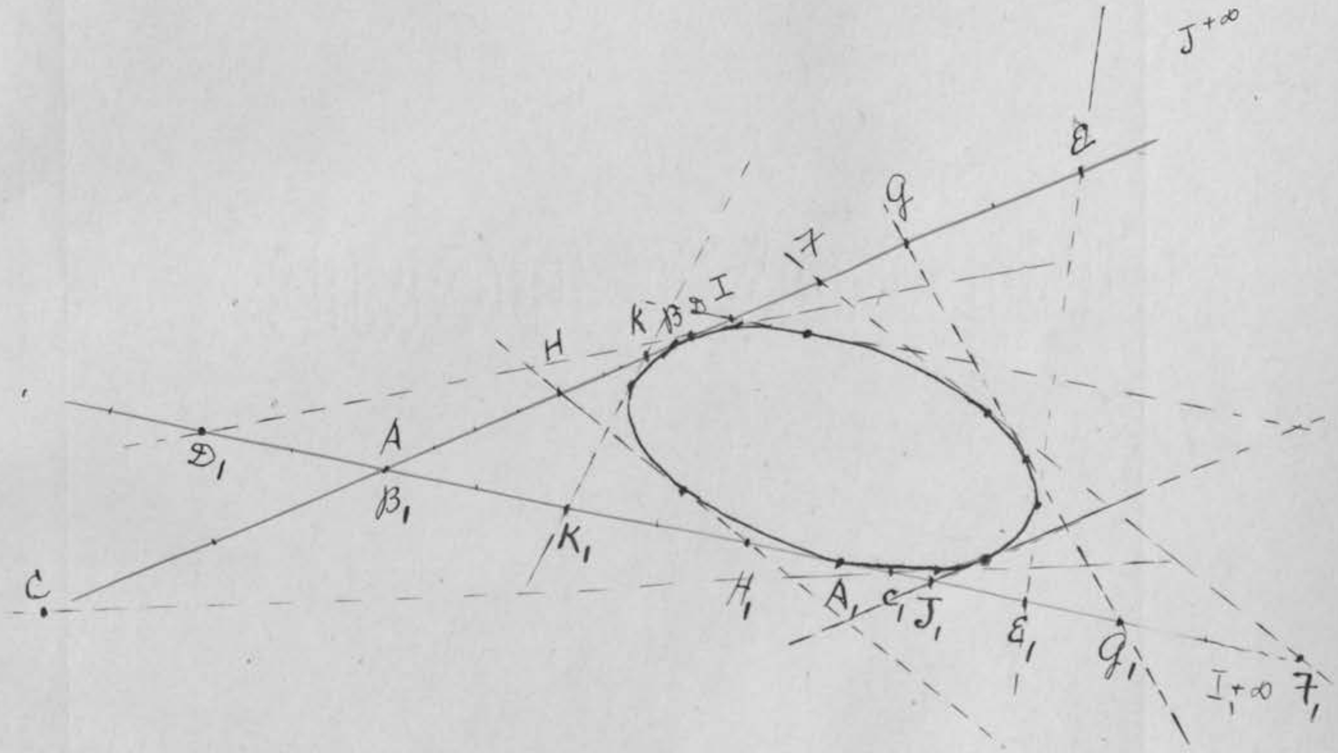


Figure XLIV.

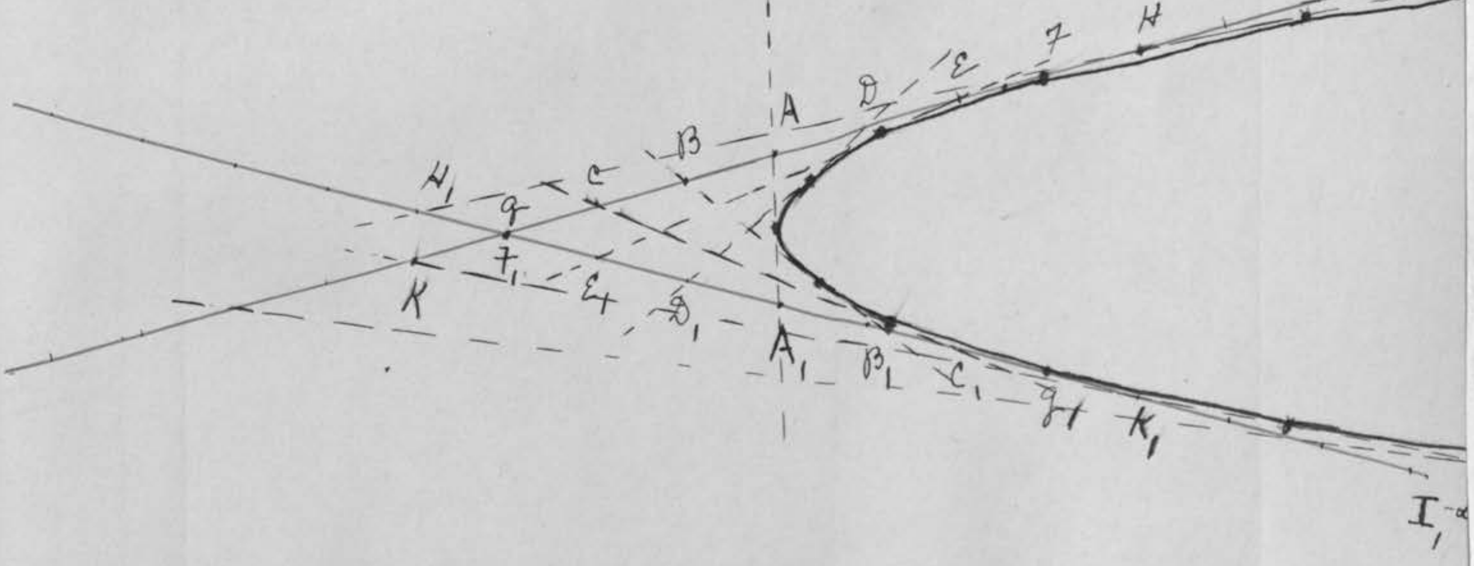


Specimen Drawing of Ellipse

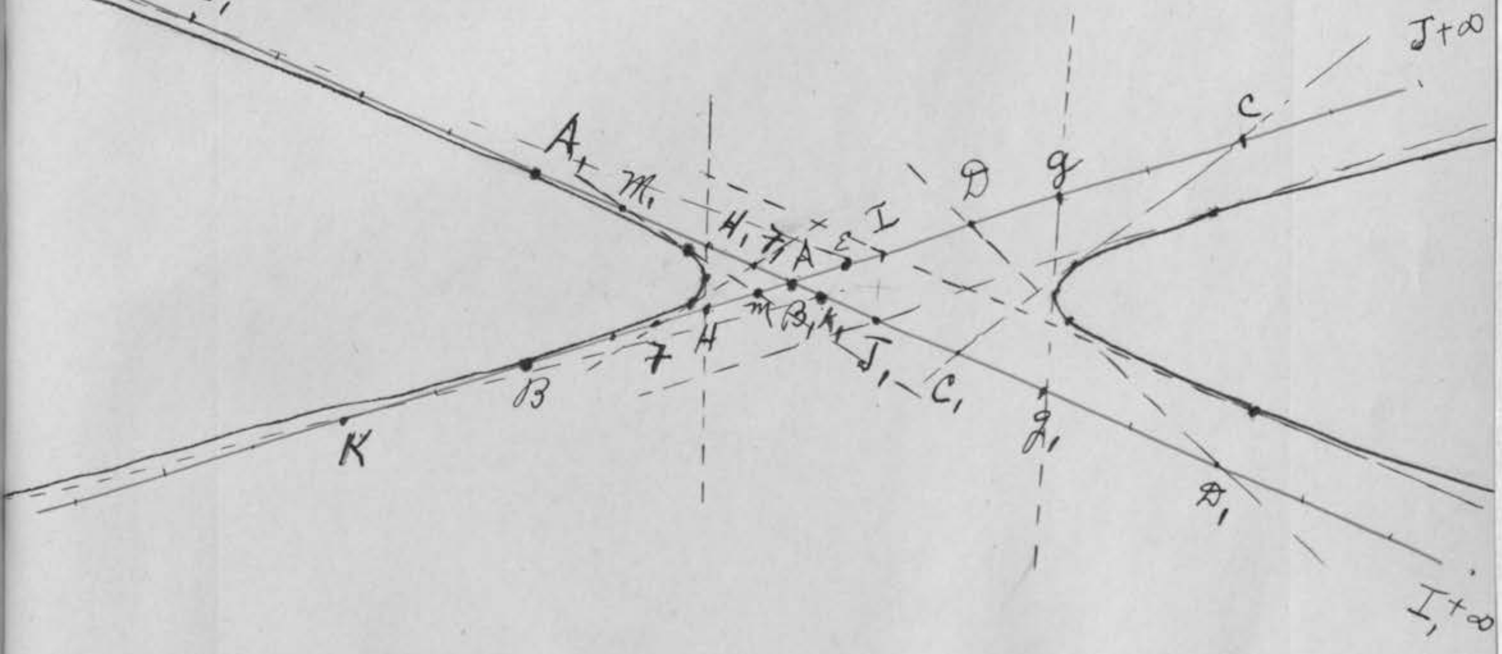


$x_1 = \frac{1}{x}$

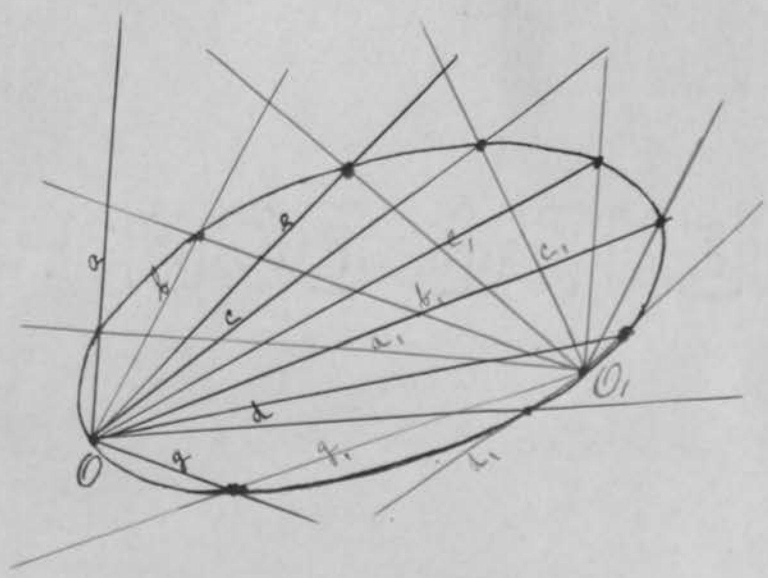
Specimen Drawing of Parabola.



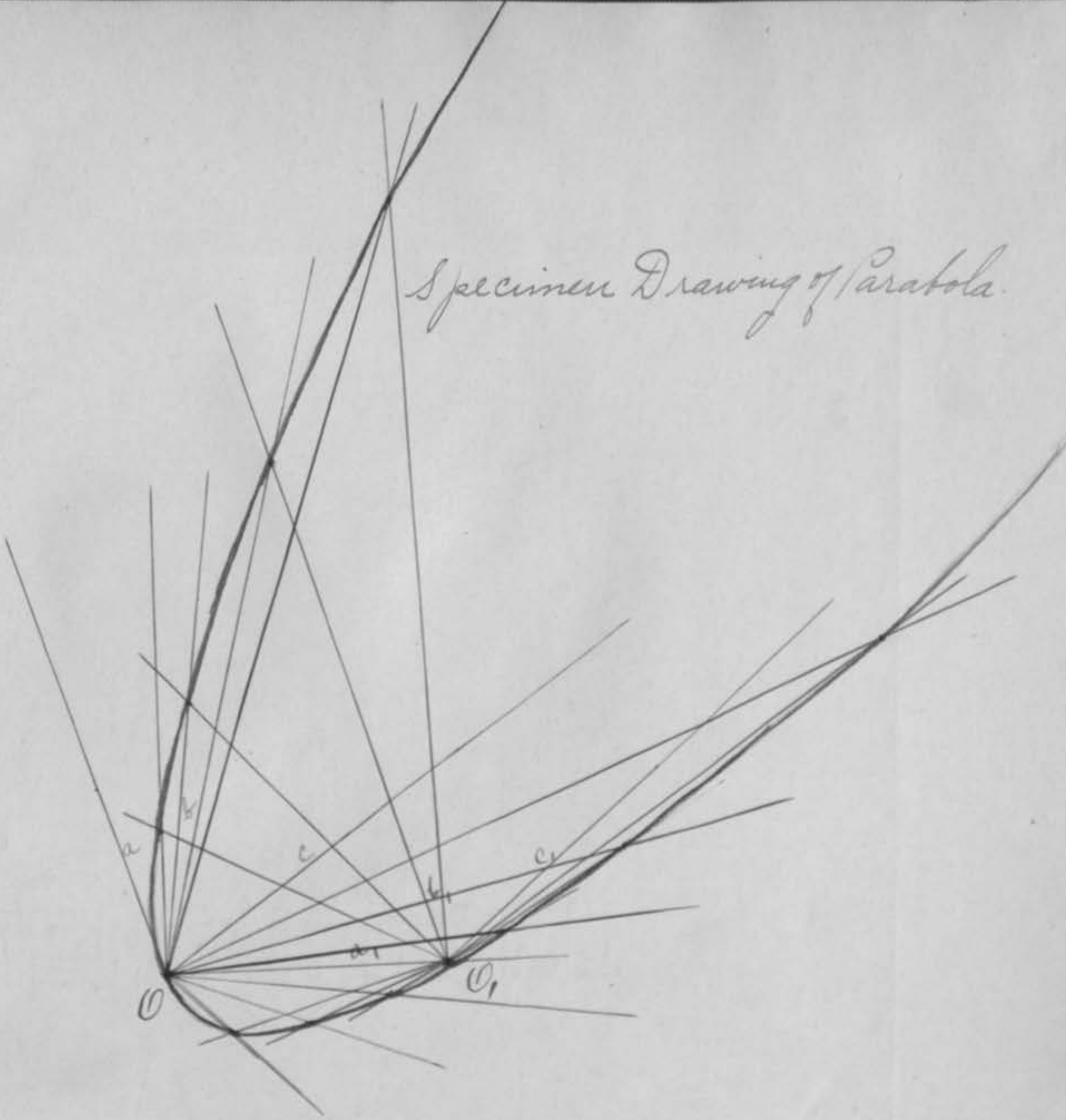
Specimen Drawing of Hyperbola.



Specimen Drawing of Ellipse.



Specimen Drawing of Parabola.



Specimen Drawing of Hyperbola.